On Polygons Inscribed within Jordan Curves

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Euler Circle IRPW, July 2023

Introduction

Figure: What is Jordan Curve?

Introduction

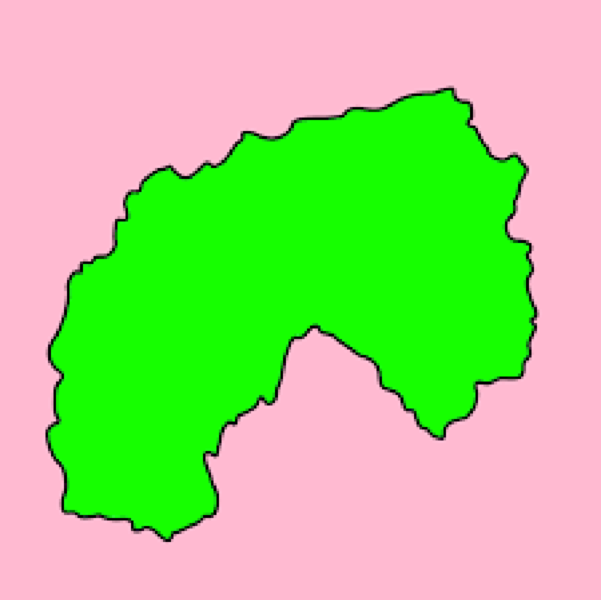
Definition of a Jordan Curve in R^2

Simple closed curve in a plane. When J is parameterized,

$$\phi(t)$$

takes the form where $\phi : [0, 1]$ in R^2 such that $\phi[0] = \phi[1]$ and ϕ is injective on [0, 1).

Introduction



Remark

Not surprisingly, a Jordan curve (J) divides a planar surface into two parts: the interior and the exterior.

The Jordan Curve Theorem

The whole set J and R^2 consist of two components: the bounded and unbounded.

Inscribing Rectangles in J

We will prove the following theorem:

Theorem (Vaughan 1977)

For some Jordan curve γ, $\exists$ some set of 4 points as vertices of a rectangle.

using a Lemma derived from topological arguments that the surface S defined as space of pair of points on $J \times J$ is homeomorphic to Mobius strip. This equivalence can be done by affine transformations.

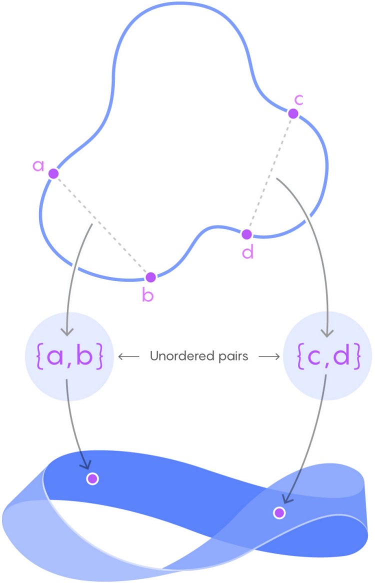
Why this homeomorphism?

Remark

To satisfy a rectangle, two distinct pair of points on J must be:

- pairwise equidistant
- share the same midpoint

Making a Möbius Strip



When plotted onto 2D space, these pairs form a Möbius strip.

Inscribing Rectangles in J

Proof. Define a new function $f : S \to R^3$ containing the image of all points above midpoint of the pairs with the z-coordinate being the distance between.

Under the assumption that f is an injection, $f(S)$ would be a Mobius strip in $x \geq 0$.

Cutting through... After gluing, $f(S) \cup In(J)$. Compactness of P^2 and Hausdorffness of R^3 allow γ to be topological projection P^2 embedded in R^3. But this is **contradiction** to a theorem of algebraic topology that **no real projective plane can be embedded into R^3.**

Inscribing Rectangles in J

What just happened?

- $A \simeq B$ means A and B are homeomorphic. If $A \simeq B$ and $B \subset C$, A **embeds** into C.
- Def. **Hausdorff** topol. space: every two distinct points can be separated by disjoint open sets.
- Def. Hausdorff space is **compact** if every open cover contains finite subcover.