On polygons inscribed by Jordan Curves

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July 18, 2023

Abstract

The question of inscribed polygons in Jordan curves has been historically attempted and certain cases were rigorously proven. I prove main theorems pertaining Jordan Curves with fundamentals of topology. Then, I discuss proofs of inscribed polygons, such as triangles, rectangles, and squares, with further generalized theorems describing thereof.

1 Introduction

1.1 History

Curves were extensively researched within the field of mathematics for centuries. During the 1870s, due to vast developments in mathematics, new definitions and revisions made to Euclidean geometry, in which the definitions of curves emerged. While initially mathematicians have attempted to define curves as algebraic solutions, this heavily restricted the types of curves that could be explored. Thus, mathematician Jordan provided a new definition of "free curves":

1.2 Definitions and Preliminaries

Definition 1.1 (Jordan curve). A Jordan curve, also denoted as J, is a simple closed curve in the plane. (In a more intuitive explanation, we can say that it is any open

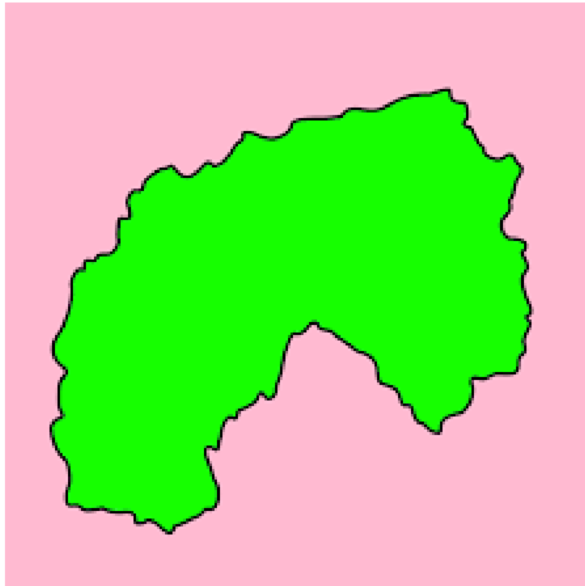
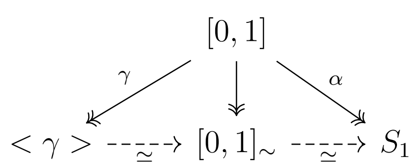


Figure 1: Explanation of Jordan curve Theorem

curve drawn such that its starting point and endpoint meet and the line never intersect itself.)

Definition 1.2 (Parameterized Jordan curve in $\mathbb{R}^2$). When J is parameterized, $\phi(t)$ takes the form where $\phi : [0, 1]$ in R^2 such that $\phi[0] = \phi[1]$ and ϕ is injective on [0, 1). This is due to homeomorphism between J and S_1.



Theorem 1.1 (Jordan Curve Theorem). The set of Jordan curve J - $\mathbb{R}^2$ consists of two distinct parts of interior region and exterior region.

This in fact allows to consider special examples of Jordan curves, such as closed fractal geometric objects or closed forms of the Weirstrauss function, in which the curve is everywhere continuous but nowhere differentiable.

Definition 1.3. A polygon is said to *inscribe* J if and only if all of its vertices are on J.

2 Inscribed Triangles

2.1 Arbitrary triangles in $\mathbb{R}^2$ (Meyerson)

Theorem 2.1 (Meyerson's Theorem). For arbitrary triangle ΔABC, vertices of similar triangles to ΔABC lie on arbitrary J. [1]

Proof. Let ΔXYZ with X on J be similar to ΔABC. Let P and Q be maximally distant points on J. We restrict ΔABC where $\angle C$ is the maximal angle. WLOG, we may assume that XY is the longest side.

Then, dilate ΔXYZ such that either Y or Z first meets J. In case of *Fig 2.*, move Z such that $Z = Q$. And during the process of moving X to satisfy $X = P$, because PQ is maximal distance within J and XY is the longest side, we see that Y must be on or outside of J. In the case where Y is outside of J, by IVT if interpreted analytically, Y is guaranteed to cross J at least once during the process of moving X towards P.

Thus Theorem 2.1 follows. □

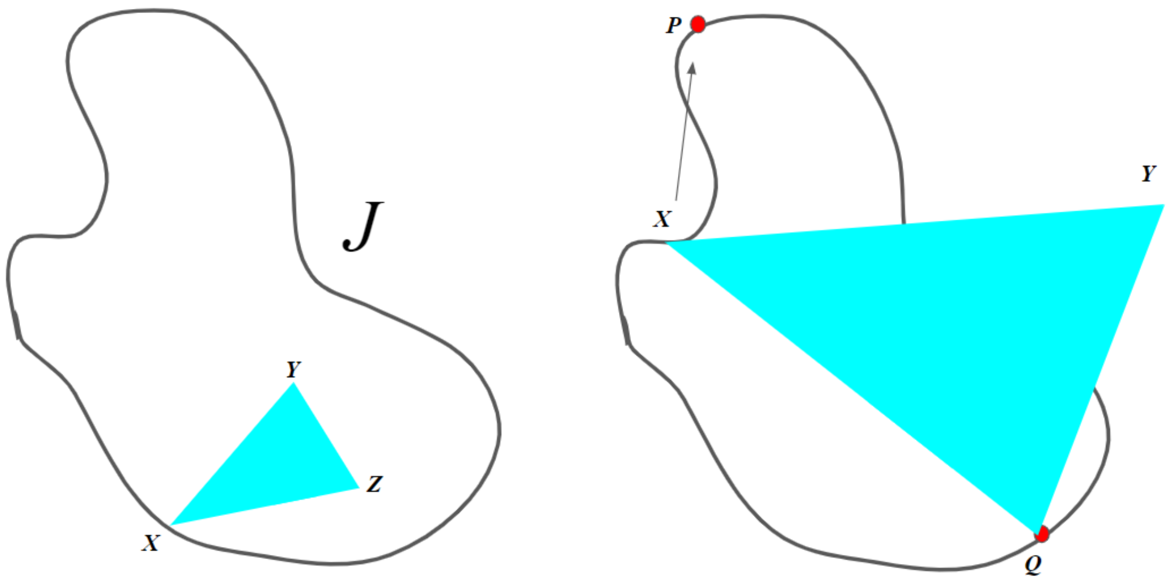


Figure 2: Explanation of Theorem 2.1

2.2 Triangles in R^n (A. Gupta and Simon Rubinstein-Salzedo)

I here summarize the results by Gupta and Rubinstein-Salzedo [2], 2021, on inscribed triangles for Jordan curves in higher dimensions of $\mathbb{R}^n$. I will use the shortened version of the proof provided by Apro [3], assuming smoothness of J.

Theorem 2.2. Let J be some Jordan curve defined in $\mathbb{R}^n$. Then there exists all vertices of a triangle $\sim \Delta ABC$ on J.

Proof. Define $\gamma : [0,1] \to \mathbb{R}^n$ be parameterized Jordan curve J. Define functions $F^x_\delta(y,z) : (x, x+\delta) \times (x, x+\delta)$ and $G^x_\delta(y,z) : (x-\delta, x) \times (x, x+\delta)$ to be angles between $\gamma(x)\gamma(y)$ and $\gamma(y)\gamma(z)$, respectively. $\Delta ABC \sim \Delta XYZ$, where X, Y, Z are arbitrary, and shared angle measures of $\theta_1, \theta_2, \theta_3$ in increasing order.

Theorem 2.3. $\limsup_{\delta \to 0+} F^x_\delta < \theta_i < \liminf_{\delta \to 0+} G^x_\delta \quad (i = 1, 2, 3 \text{ and } 0 < x < 1)$, then $\delta(x)$ is the point where lies the vertex of some triangle ΔXYZ with angle θ_i at that point.

Lemma 2.1. Assuming that there is a point $\gamma(s)$ for which no such t exists, then for any $t_1, t_2 \in [0,1] - s \; \exists \; I_{t_1}(\gamma) \simeq I_{t_2}(\gamma)$ in $\mathbb{R}^n - S$.

This is due to the homotopic property of $H_{t_1,t_2}(\cdot, T) = I_{(1-T)t_2+(T)t_1}(\gamma)$.

Assume that some $s \in [0, 1)$ satisfies Lemma 5.4. If we choose t_1 such that $\|\gamma(t_1)$ - $\gamma(s)\|$ is maximal, $I_{t_2}(\gamma)$ is homotopic to path at $\gamma(s)$ by straight line homotopy.

For some $\delta > 0$ such that $|\mathbb{R}^{n-1} \cap I_{s+\delta}(\gamma)| \geq 2$, $I_{s+\delta} \not\simeq I_{t_1}(\gamma)$.

This contradicts Lemma 2.1 and completes the proof.

□

3 Inscribed Quadrilaterals

3.1 Rectangles (Vaughan 1977)

Theorem 3.1 (Vaughan's Theorem). For some Jordan curve γ, $\exists$ some set of 4 points as vertices of a rectangle.

To prove this theorem, we use the following fact on the existence of rectangles. To satisfy a rectangle, two distinct pair of points on J must be:

- pairwise equidistant
- share the same midpoint

I omit the topological arguments that lead to the following crucial lemma:

Lemma 3.2. The surface S defined as space of pair of points on $J \times J$ is homeomorphic to Mobius strip. This equivalence can be done by affine transformations.

Proof. Define a new function $f : S \to R^3$ containing the image of all points above midpoint of the pairs with the z-coordinate being the distance between. Under the assumption that f is an injection, $f(S)$ would be a Mobius strip in $x \geq 0$. After gluing, $f(S) \cup In(J)$. Compactness of $\mathbb{P}^2$ and Hausdorffness of $\mathbb{R}^3$ allow γ to be topological projection P^2 embedded in R^3. But this is **contradiction** to a theorem of algebraic topology that **no real projective plane can be embedded into** R^3. Thus there exists two distinct pair of points mapping to the same point on J.

□

Vaughan's original argument. Vaughan relies on the fact that there is no continuous embedding of the Klein bottle into $\mathbb{R}^3$.

Theorem 3.2. There is no continuous embedding of the Klein bottle into $\mathbb{R}^3$.

Proof. The brief sketch of the proof is as follows. If there exists such an embedding, there also exists an embedding into the 3-Sphere, or S_3. $(K \subset S^3)$
However, Alexander duality states that

$$Z/2 \simeq H^2(K) \simeq \widetilde{H_0}(S^3 - K) \simeq Z^r$$

for some r, which leads to a contradiction. □

Considering the images made by the map of ordered pairs to encoded midpoint of the segment and lengths, Vaughan concluded that the union consists of two Mobius bands and thus a Klein bottle. And because the Klein bottle K is embedded, it contradicts the nonexistence of an embedded Klein bottle. □

3.2 Rhombi (M. J. Nielsen)

Nielsen [4] uses a powerful theorem known as the mountain climbing problem proven by Homma.

For proving the case of rhombi, let us consider piecewise linear version of the problem.

Theorem 3.3 (Polygonal mountain climbing problem). Let $f, g : [0,1] \to [0,1]$ be piecewise linear continuous functions with $f(0) = g(0) = 0$ and $f(1) = g(1) = 1$. Then there exist piecewise linear continuous functions $r, s : [0,1] \to [0,1]$ with $r(0) = s(0) = 0$ and $r(1) = s(1) = 1$ such that $f \circ r = g \circ s$.

Theorem 3.4 For polygonal jordan curve J and any line l in the plane, J inscribes a rhombus with two sides parallel to l.

Proof. Assume, WLOG, that the line segment l is parallel to the x-axis, and let z_{min} and z_{max} be two points on J with y-coordinates 0 and 1, respectively. These points divide J into two arcs, denoted as A_1 and A_2.

The theorem can be restated as follows: There exist piecewise-linear continuous functions x_1, x_2, and $y : [0, 1] \to \mathbb{R}$ such that:

$(x_i(0), y(0)) = z_{min}, (x_i(1), y(1)) = z_{max}$, and $(x_i(t), y(t)) \in A_1$ for all $t \in [1, 2]$ and $i \in 1, 2$.

Let $\gamma_i(t) = (u_i(t), v_i(t))$ be parametrizations of arcs A_i from z_{min} to z_{max} for $i = 1, 2$. Since v1 and v2 are piecewise linear and continuous, by theorem 6.4, there exist piecewise-linear continuous functions r and s: [0, 1] → R with r(0) = s(0) = 0 and r(1) = s(1) = 1, such that $v_1 \circ$ r = $v_2 \circ$ s. Thus, we can define the following functions:

$$x_1(t) = u_1(r(t)), x_2(t) = u_2(s(t)), y(t) = v_1(r(t)).$$

These functions satisfy the conditions of the theorem. □

3.3 Square (open problem)

The yet unsolved square inscribing problem was first conjectured by Toeplitz [5]:

Conjecture 3.1. All planar J contains vertices of a square.

While the variations of this conjecture has been proven for special cases, the generalization is unsolved. I aim to compile the progress and results of the field.

Theorem 3.5. (The case of smooth curves, Schnirelmann). When J is a C^2-continuous closed curve, J contains the vertices of a square.

Theorem 3.6. (Strenghtened theorem, W. Stromquist). Let J be a Jordan curve on which every point p has a neighborhood A_p and a direction dp such that no chord of J is contained in A_p and parallel to dp. Then J contains the vertices of a square.

Stromquist's theorem can cover C_1 curves, convex curves, and polygons, which satisfy the conditions.

Theorem 3.7. (The case of convex curves, A. Emch). Let J be a convex Jordan curve in the plane. Then J contains the vertices of a square.

4 Applications

While topology itself already has great amount of real-world applications, Jordan curves were specifically used to analyze robotic tragectories [5], digital image processing [6], and GPS Navigation systems.

Acknowledgement

I express deep gratitude for my mentor Dr. Simon Rubinstein-Salzedo and TA Alphonse Houdalga at Stanford University for supervising me throughout the completion of the above paper.

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