Hyperbolic Space

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Euclid's Postulates

1. Between any two points, a line can be drawn
2. Any finite line can be extended into an infinite one
3. For any point and distance, a circle can be constructed with that point as it's center and radius equal to the distance
4. All right angles are congruent
5. Given two lines a and b in the plane and a third line l which intersects both lines, if the two interior angles on one side of l sum to less than two right angles, then if a and b are extended indefinitely, they will meet on the side where the sum of angles is less than two right angles.

The 5th postulate is notably more complex than the others. There are some equivalent statements which are simpler, such as the Playfair postulate, which states that for any line and point not on that line, there exists a unique line which passes through that point which is does not intersect that line.

Even with such developments, mathmaticians still continued to try and derive the 5th postulate from the other 4.
However, it turns out this is impossible, as there are consistent systems of geometry which do not satisfy the 5th postulate. One of these is **Hyperbolic Geometry**

Hyperbolic Space can be modeled with the Upper Half plane, denoted $\mathbb{H}$ which is the set of points in the complex plane with positive imaginary component. There are other models of Hyperbolic Space, but we will mainly look at the Upper Half Plane Model.

Before we fully define the Upper Half Plane Model, we must first outline objects known as **Metric Spaces**. A metric space is a set X with a mapping $d : X \times X \to \mathbb{R}$ called a metric such that for all $x, y, z \in X$

1. $d(x, x) = 0$, and $d(x, y) > 0$ when $x \neq y$
2. $d(x, y) = d(y, x)$
3. $d(x, y) \leq d(x, z) + d(y, z)$

In Euclidean Space, the length of a path $\sigma\colon [0,1] \to \mathbb{R}^2$ is defined as

$$\int_0^1 |\sigma'(t)| dt$$

.

However, in Hyperbolic Space, the length of a path $\sigma\colon [0,1] \to \mathbb{H}$ is instead

$$\int_0^1 \frac{1}{\Im(\sigma(t))} |\sigma'(t)| dt$$

Where $\Im(z)$ is the imaginary component of z.

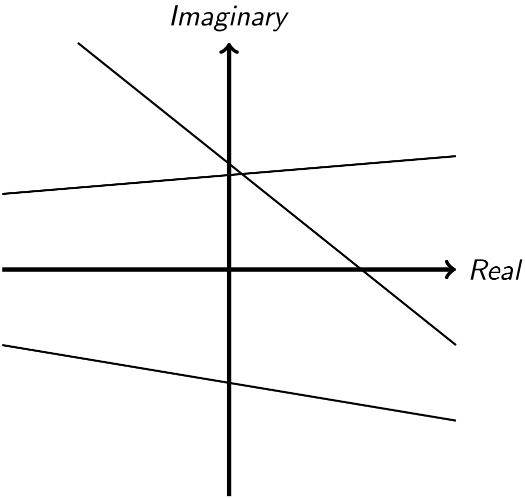
In both Euclidean and Hyperbolic space, we consider the distance between two points to be the length of the path between the two points which has minimum length when compared to all other paths between those two points. The path which has minimal length is considered to be a geodesic.

More formally, for any two points $a, b \in \mathbb{H}$ we define

$$d(a,b) = \inf\{\int_0^1 \frac{1}{\Im(\sigma(t))}|\sigma'(t)|dt \colon \sigma \text{ is a path from a to b}\}$$

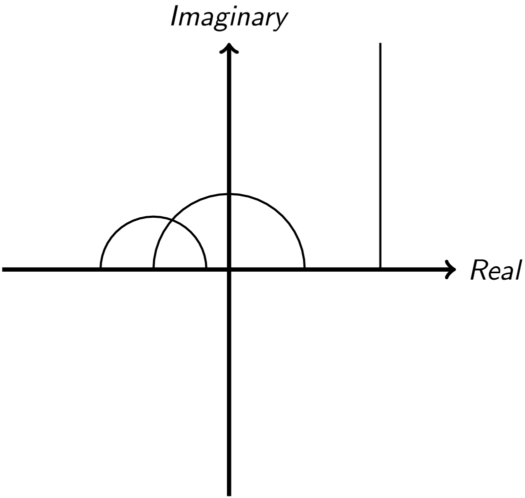
One can verify that this is indeed a metric.

In Euclidean Space, geodesics look like this.



They are straight lines

However, geodesics in Hyperbolic space instead look like this



They are either vertical lines or circles centered on the real axis.

The above image also shows how Hyperbolic Space violates the Playfair postulate, as there are multiple lines going through a single point which do not intersect a third line, which implies that Hyperbolic Geometry violates the 5th postulate. One can also verify that Hyperbolic geometry satisfies the other 4 axioms.

Isometries are mappings from a space to itself which leave the distance between points unchanged. More formally, if d_X is a metric on X and d_Y is a metric on Y, then $f : X \to Y$ is an isometry if for any $x, y \in X$, $d_X(x, y) = d_Y(f(x), f(y))$

In familiar 2D Euclidean Space, some isomtries include translations

$$f(z) = z + k, k \in \mathbb{C},$$

and rotations

$$f(z) = ze^{i\theta}, \theta \in \mathbb{R}.$$

Isometries of the Upper Half plane are more complex. They take the form

$$f(z) = \frac{az+b}{cz+d}, \ a, b, c, d \in \mathbb{R}, \ ad - bc > 0$$

. These are called Möbius Transforms.
We will show that these are isometries.

Let $\sigma(t)$ be a path from z to w, and let $\gamma(z) = \frac{az+b}{cz+d}$ be a Möbius Transform. Then, it follows that

$$|\gamma'(z)| = \frac{ad - bc}{|cz + d|^2}$$

and that

$$\Im(\gamma(z)) = \frac{ad - bc}{|cz + d|^2} \Im(z)$$

Thus, the length of $\gamma(\sigma(t))$ is

$$\begin{aligned}
&\int_0^1 \frac{1}{\Im(\gamma(\sigma(t)))}|(\gamma \circ \sigma)'(t)|dt \\
&= \int_0^1 \frac{|c\sigma(t)+d|^2}{(ad-bc)\Im(\sigma(t))}|\gamma'(\sigma(t))||\sigma'(t)|dt \\
&= \int_0^1 \frac{|c\sigma(t)+d|^2|}{(ad-bc)\Im(\sigma(t))}|\frac{ad-bc}{|c\sigma(t)+d|^2}||\sigma'(t)|dt \\
&= \int_0^1 \frac{1}{\Im(\sigma(t))}|\sigma'(t)|dt
\end{aligned} \tag{1}$$

Which is the same as the length of $\sigma(t)$. Thus, because Möbius Transforms preserve the lengths of paths, they must preserve the distance between points. Thus, Möbius Transforms are isometries of $\mathbb{H}$.

One consequence of the fact that Möbius Transforms are isometries are the fact that Möbius transforms send geodesics in $\mathbb{H}$ to other geodesics. Assume γ is a Möbius Transform and σ is a path from a to b but $\gamma(\sigma(t))$ is not a geodesic from $\gamma(a)$ to $\gamma(b)$. Then, there exists a path from $\gamma(a)$ to $\gamma(b)$, let us call it $\phi(t)$, which is shorter than $\gamma(\sigma(t))$.
However, it turns out the inverse of a Möbius Transform is still a Möbius transform. Since $\gamma^{-1}(t)$ is a Möbius transform, and Möbius transforms preserve length, it follows that the length of $\gamma^{-1}(\phi(t))$ is less than $\sigma(t)$. however, since $\gamma^{-1}(\phi(t))$ is a path from a to b, this violates the assumption that $\sigma(t)$ is a geodesic. Thus, $\phi(t)$ cannot exist, and thus $\gamma(\sigma(t))$ is a geodesic.

Möbius Transforms have many other properties. For example, as previously stated, the inverse of a Möbius transform is still a Möbius Transform. Also, the composition of two Möbius transforms is another Möbius Transform. Furthermore, Möbius transforms map vertical lines and circles with real centers to other vertical lines and circles with real centers. Lastly, for any two points, there is a Möbius transform which maps two points on the imaginary axis to those two points.

Lastly, before we state the Gauss-Bonnet theorem, we must define angle and area in $\mathbb{H}$. The area of some subset $A \subseteq \mathbb{H}$ is

$$\int\int_A \frac{1}{\Im(z)^2} dz$$

The angle between two paths $\sigma(t)$ and $\phi(t)$ which intersect at $t = 0$ is defined as follows: $\cos^{-1}\left(\frac{\langle \phi'(0), \sigma'(0) \rangle}{|\phi'(0)||\sigma'(0)|}\right)$, where $\langle \phi'(0), \sigma'(0) \rangle$ is the dot product of the two vectors.

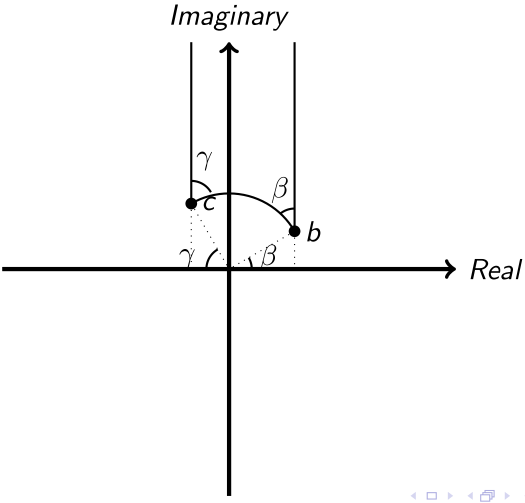
As one might suspect, Area and Angle are both invariant under Möbius Transform. I won't prove this here, but you can find proofs in my expository paper.

Hyperbolic space has many properties which are not found in Euclidean space. For example, the area of a triangle is related to it's internal angles. In Euclidean space, the area and internal angles of a triangle are unrelated. However, in hyperbolic space, the area of a triangle is equal to

$$\pi - \alpha - \beta - \gamma$$

, where α, β, γ are the internal angles of the triangle. This is known as the Gauss-Bonnet theorem.

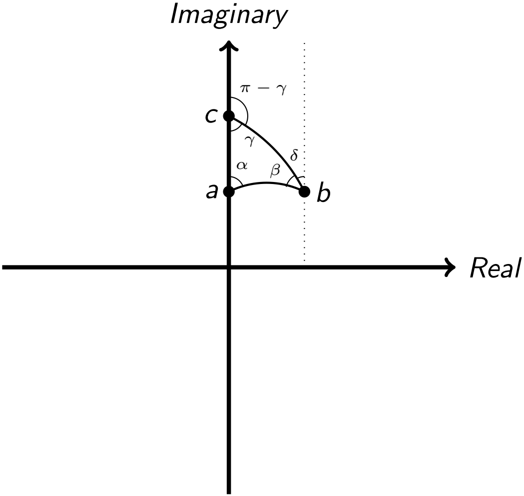
To prove the theorem, we first prove it for a triangle with one vertex at ∞ For such a triangle, we can transform it with horizontal translations and scaling by a constant factor until one edge of the triangle lies on the unit circle.



Then, we calculate the area of the triangle by integration

$$\begin{aligned} Area(\triangle) \\ &= \int_{\Re(c)}^{\Re(b)} \int_{\sqrt{1-x^2}}^{\infty} \frac{1}{y^2} dydx \\ &= \int_{\Re(c)}^{\Re(b)} \frac{1}{\sqrt{1-x^2}} dx \\ &= \int_{\pi-\gamma}^{\beta} -1 d\theta \text{ (After substituting } x = cos(\theta)) \\ &= \pi - \beta - \gamma \end{aligned} \tag{2}$$

Then, we show that all triangle areas are the difference of two triangles with vertex at ∞



We see that the $Area(\triangle abc) = Area(\triangle ab\infty) - Area(\triangle bc\infty)$ which then by the previous result implies that

$Area(\triangle abc) = \pi - \alpha - \beta - \delta - (\pi - \pi + \gamma - \delta) = \pi - \alpha - \beta - \gamma$

This proves the theorem.

This theorem implies many interesting things. For example, in hyperbolic geometry, triangles have a maximum area. That occurs when α, β, γ are all 0, in which case the triangle has an area of π. Also, in contrast to euclidean geometry, the sum of angles in a hyperbolic triangle is not necessarily π, but must be less than π.