THE GAUSS-BONNET THEOREM FOR REGULAR POLYGONS IN THE HYPERBOLIC PLANE

JUNLIN JULIAN JIANG

1. Abstract

This paper is an exploration of hyperbolic space, specifically the model of hyperbolic space known as the Upper Half Plane, it's associated group of isometries known as the Möbius group, and finally the Gauss-Bonnet theorem, which relates the area and internal angles of triangles in hyperbolic space.

2. Introduction

For several Millennia, Euclid's postulates, outlined in the book Euclid's elements was considered to be the foundation of geometry. These axioms are, in order

(1) Between any two points, a line can be drawn
(2) Any finite line can be extended into an infinite one
(3) For any point and distance, a circle can be constructed with that point as it's center and radius equal to the distance
(4) All right angles are congruent
(5) Given two lines a and b in the plane and a third line l which intersects both lines, if the two interior angles on one side of l sum to less than two right angles, then if a and b are extended indefinitely, they will meet on the side where the sum of angles is less than two right angles.

One of these 5 postulates is not like the other. While the first 4 postulates can all be stated simply, the 5th postulate is more complex. Many mathematicians have tried, and failed to derive the 5th postulate from the other 4. As it turns out, this is because it is not possible. However, these efforts eventually led to the development of hyperbolic geometry, a type of geometry which obeys the first four postulates, but not the 5th.

The study of hyperbolic geometry was first pioneered by János Bolyai , Carl Friedrich Gauss, and Nikolai Lobachevsky. Although at first many suspected that hyperbolic geometry was impossible and inconsistent, it was eventually proven that any inconsistency in hyperbolic geometry would lead to an inconsistency in euclidean geometry. Thus, as most people believe euclidean geometry to be self-consistent, it follows that hyperbolic geometry is as well.

The main purpose of this paper will be to build up to the Gauss-Bonnet theorem. This theorem states that for any triangle in hyperbolic space, it's area is equal to π minus the sum of the triangle's internal angles. This is in stark contrast to euclidean geometry, where the sum of the interior angles of a triangle is always $/pi$.

To build up to this theorem, we will start by defining the model of hyperbolic space known as the Upper Half Plane, as well as defining the metric on this space. In the next section,

Date: July 15, 2023.

we will explore Möbius Transforms due to their role as the isometry group of the Upper Half Plane, and outline some useful properties which these transformations have. After that, we will describe the geodesics of the Upper Half Plane, as well as angle and area in hyperbolic space, which will be necessary to even state the Gauss-Bonnet theorem. Finally, in the last section, we will prove the Gauss-Bonnet theorem.

3. Acknowledgments

The author would like to thank Dr. Hvidsten and Dr. Walkden for their extremely helpful resources on hyperbolic geometry. The author would also like to thank Dr. Rubinstein-Salzedo for organizing Eulercircle and providing this opportunity to the author, as well as Leonardo Bonanno for being a TA and helping with the paper.

4. Body

5. Distance in Hyperbolic Space

The model of hyperbolic space which will be used in this paper is the *Upper Half Plane* model.

Definition 1. *The Upper Half Plane is the set* $\mathbb{H} := \{z \in \mathbb{C} | \Im(z) > 0\}$*, where* $\Im(z)$ *is the imaginary component of* z*.*

What distinguishes the Upper Half Plane model from regular Euclidean space is a different notion of distance.

Definition 2. *A path in the Upper Half Plane is the image of a continuous function* $\sigma\colon [0,1] \to \mathbb{H}$*. Such a path is said to be piece-wise continuously differentiable if all but finitely many* $t \in [0,1]$*,* $\sigma(t)$ *is differentiable the function* $\sigma'(t)$ *is continuous.*

Definition 3. *The length of such a piece-wise continuously differentiable path in the Upper Half Plane* $\sigma(t)$ *is defined as follows: If* $\sigma(t)$ *is differentiable at all points except* $t_m \in [0,1]$ *for* $1 \leq m \leq n$ *for some* $n \in \mathbb{N}$*, then, the length of said path is*

$$Length_{\mathbb{H}}(\sigma(t)) := \sum_{i=0}^{n} \int_{t_i}^{t_{i+1}} \frac{1}{\Im(\sigma(t))} |\sigma'(t)| dt.$$

Intuitively, the length of a path in $\mathbb{H}$ is the computed by integrating $\frac{1}{\Im(\sigma t)}|\sigma'(t)|$ over each stretch of the path where the path is differentiable.

This is in contrast to Euclidean space, where the length of a path is computed by integrating $|\sigma'(t)|$ over the path

From the definition of path length, we can define a *metric* on $\mathbb{H}$. A metric is just a notion of distance, which is required to satisfy certain properties.

Definition 4. *A metric on a set* X *is a function* $d\colon X \times X \to \mathbb{R}$ *such that for any* $x, y, z \in X$*:*

- $d(x,y) > 0$ *when* $x \neq y$ *and* $d(x,x) = 0$
- $d(x,y) = d(y,x)$
- $d(x,y) \leq d(x,z) + d(y,z)$

A set together with a metric is called a metric space. For example, one can verify that $\mathbb{R}$ together with the function $d(x,y) = |x - y|$ together form a metric space.

We can define metric on $\mathbb{H}$ via the notion of path length.

Definition 5. *For all $x, y \in \mathbb{H}$*

$$d_{\mathbb{H}}(x,y) := \inf\{Length_{\mathbb{H}}(\sigma) | \sigma \text{ is a piece wise continuously differentiable path from } x \text{ to } y\}.$$

The infimum of a set A, denote $\inf(A)$, is the *largest* real number $r \in \mathbb{R}$ such that for all $a \in A$, $r \leq a$. In plain language, it is the largest number which is still smaller than or equal to all numbers in the set.

One can verify that this satisfies all 3 axioms, and is thus a metric. The explicit form of the metric is

$$d_{\mathbb{H}}(z,w) = \cosh^{-1}(1 + \frac{|z-w|^2}{2\Im(z)\Im(w)}).$$

This will not be proved now, as it requires first proving some additional properties of $\mathbb{H}$

Regarding metric spaces, there are 2 other terms which we must define

Definition 6. *A isometry of a metric space X is a bijection $f\colon X \to X$ such that for all $x, y \in X$, $d(x,y) = d(f(x), f(y))$*

In other words, an isometry of a metric space is just a transformation which preserves distances between points in a metric space. For example, a translation is an isometry of the metric space $\mathbb{R}$ equipped with the metric $d(x,y) = |x-y|$ because it preserves the distance between points.

Definition 7. *A geodesic between two points x, y in a metric space X is a piecewise differentiable path γ from x to y such that for any other path σ between x and y, $Length(\gamma) \leq Length(\sigma)$*

Note that in general, between two points in a metric space, there may exist no geodesics or many geodesics. However, in $\mathbb{H}$, we will prove that between any two points, there exists exactly one geodesic.

6. Möbius Transforms and Isometries of $\mathbb{H}$

Definition 8. *A Möbius transform is a function*

$$\begin{array}{rcl} f\colon \mathbb{C}\cup\{\infty\} & \to & \mathbb{C}\cup\{\infty\}, \\ z & \to & \frac{az+b}{cz+d} \end{array}$$

where $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$. The set of all Möbius transforms is denoted $M\ddot{o}b(\mathbb{H})$.

Some examples of Möbius transforms include $f(z) = z$, $f(z) = -\frac{1}{z}$ and $f(z) = z+1$

The set $\mathbb{C}\cup\{\infty\}$ is the complex plane equipped with a point at infinity in order to make the Möbius transform a function. We say that for some Möbius transform $\gamma(z) = \frac{az+b}{cz+d}$,

$$\gamma(\infty) = \frac{a + \frac{b}{\infty}}{c + \frac{d}{\infty}} = \frac{a+0}{c+0} = \frac{a}{c},$$

and also that

$$\gamma(-\frac{d}{c}) = \frac{-\frac{ad}{c} + b}{c\frac{-d}{c} + d} = \frac{b - \frac{ad}{c}}{0} = \infty.$$

Depending on the specific situation we shall either consider the effect of Möbius transforms on $\mathbb{C}\cup\{\infty\}$ or on $\mathbb{H}$.

Note that it is very natural to drop the restriction that a, b, c, d are real and allow a, b, c, d to be any complex number, as well as loosen the restriction that $ad - bc > 0$ to $ad - bc \neq 0$.

However, when imposing the above restrictions, this ensures that Möbius transforms are bijections from the Upper Half Plane to itself, which is a fact we shall soon prove.

Möbius transforms have certain useful properties which make them nice. For example, all Möbius transforms have inverses which are also Möbius transforms.

Theorem 6.1. *All Möbius transforms have inverses which are also Möbius transforms.*

Proof. Let $\gamma \in M\ddot{o}b(\mathbb{H})$. Then, $\gamma(z) = \frac{az+b}{cz+d}$ where $ad - bc > 0$ and $a, b, c, d \in \mathbb{R}$. Let $\phi(z) = \frac{dz-b}{-cz+a}$. Firstly, $\phi \in M\ddot{o}b(\mathbb{H})$ because $da - (-b)(-c) = ad - bc > 0$. Next,

$$\phi(\gamma(z)) = \frac{d\frac{az+b}{cz+d} - b}{-c\frac{az+b}{cz+d} + a} = \frac{daz + bd - bcz - bd}{-caz - bc + acz + da} = \frac{adz - bcz}{ad - bc} = z.$$

Finally,

$$\gamma(\phi(z)) = \frac{a\frac{dz-b}{-cz+a} + b}{c\frac{dz-b}{-cz+a} + d} = \frac{adz - ab - bcz + ab}{cdz - bc - cdz + ad} = \frac{adz - bcz}{ad - bc} = z.$$

Because $\gamma(\phi(z)) = \phi(\gamma(z)) = z$, that implies $\gamma^{-1}(z) = \phi(z) \in M\ddot{o}b(\mathbb{H})$. Thus, all Möbius transforms have inverses which are also mobius transforms

□

In addition to the existence of inverses which are also Möbius transforms, we can also show that the composition of two Möbius transforms is still a Möbius transforms.

Theorem 6.2. *The composition of two Möbius transforms is still a Möbius transform.*

Proof. Let $\gamma_1, \gamma_2 \in M\ddot{o}b(\mathbb{H})$. Then, $\gamma_1(z) = \frac{az+b}{cz+d}$ and $\gamma_2(z) = \frac{ez+f}{gz+h}$ for some $a, b, c, d, e, f, g, h \in \mathbb{R}$ and $ad - bc > 0$ and $eh - fg > 0$. Then,

$$\begin{aligned} \gamma_1(\gamma_2(z)) &= \frac{a\frac{ez+f}{gz+h} + b}{c\frac{ez+f}{gz+h} + d} \\ &= \frac{aez + af + bgz + bh}{cez + cf + gdz + dh} \\ &= \frac{(ae + bg)z + (af + bh)}{(ce + gd)z + (cf + dh)} \end{aligned} \tag{1}$$

Since $(ae + bg), (af + bh), (ce + gd), (cf + dh) \in \mathbb{R}$, all that remains is to show that

$$(ae + bg)(cf + dh) - (af + bh)(ce + gd) > 0.$$

This takes a little bit of computation.

$$\begin{aligned} (ae + bg)(cf + dh) - (af + bh)(ce + gd) &= aecf + bgcf + aedh + bgdh - afce - bhce - bhgd - afgd \\ &= bgcf + aedh - bhce - afgd \\ &= bc(gf - he) + ad(eh - fg) \\ &= (ad - bc)(eh - fg) \end{aligned} \tag{2}$$

And since $ad - bc > 0$ and $eh - fg > 0$, we see that $(ad + bg)(cf + dh) - (af + bh)(ce + gd) = (ad - bc)(eh - fg) > 0$. Thus, $\gamma_1(\gamma_2(z))$ is a Möbius transform □

For those who know group theory, you can verify that $M\ddot{o}b(\mathbb{H})$ is a group under composition of functions.

Möbius transforms are of relevance to hyperbolic geometry because $M\ddot{o}b(\mathbb{H})$ is the set of orientation preserving isometries of $\mathbb{H}$. To show this, we must first show that a Möbius transform is a bijection from $\mathbb{H}$ to $\mathbb{H}$.

Lemma 6.3. *Let $\gamma(z) = \frac{az+b}{cz+d} \in M\ddot{o}b(\mathbb{H})$. Then, γ is a bijection from $\mathbb{H}$ to $\mathbb{H}$.*

Proof. Firstly, let $z \in \mathbb{H}$. If $z \in \mathbb{H}$, $\Im(z) > 0$. Now, consider the value of $\Im(\gamma(z))$.

$$\begin{aligned}
\Im(\gamma(z)) &= \frac{\gamma(z) - \overline{\gamma(z)}}{2i} \\
&= \frac{az+b}{2i(cz+d)} - \frac{a\overline{z}+b}{2i(c\overline{z}+d)} \\
&= \frac{(az+b)(c\overline{z}+d) - (cz+d)(a\overline{z}+b)}{2i(cz+d)(c\overline{z}+d)} \\
&= \frac{acz\overline{z} + bc\overline{z} + adz + bd - acz\overline{z} - ad\overline{z} - bcz - bc}{2i|cz+d|^2} \\
&= \frac{ad(z-\overline{z}) - bc(z-\overline{z})}{2i|cz+d|^2} \\
&= \frac{ad-bc}{|cz+d|^2}\Im(z)
\end{aligned} \tag{3}$$

Because $\gamma \in M\ddot{o}b(\mathbb{H})$, $ad - bc > 0$. This implies that $\Im(\gamma(z)) = \frac{ad-bc}{|cz+d|^2}\Im(z) > 0$. Thus, $\gamma(z) \in \mathbb{H}$. This implies that

$$\begin{aligned} \gamma \colon \mathbb{H} &\to \mathbb{H} \ . \\ z &\to \gamma(z) \end{aligned}$$

Also, by Theorem 6.1,

$$\begin{aligned} \gamma^{-1} \colon \mathbb{H} &\to \mathbb{H} \ . \\ z &\to \gamma(z) \end{aligned}$$

This implies that γ is a bijection from $\mathbb{H}$ to $\mathbb{H}$ for all $\gamma \in M\ddot{o}b(\mathbb{H})$

□

Note that although Möbius transforms are bijections from $\mathbb{H}$ to $\mathbb{H}$.

Theorem 6.4. *Möbius transforms are isometries of $\mathbb{H}$.*

Proof. Let $x, y \in \mathbb{H}$ and $\gamma(z) = \frac{az+b}{cz+d} \in M\ddot{o}b(\mathbb{H})$ and assume that $d_{\mathbb{H}}(x,y) \neq d_{\mathbb{H}}(\gamma(x),\gamma(y))$. Then, without loss of generality, assume that $d_{\mathbb{H}}(x,y) < d_{\mathbb{H}}(\gamma(x),\gamma(y))$.

Recall that from Definition 5,

$$d_{\mathbb{H}}(x,y) := \inf\{Length_{\mathbb{H}}(\sigma) | \sigma \text{is a piece wise continuously differentiable path from x to y}\}.$$

From this definition we see that there must exist a piece wise differentiable path σ from x to y such that $d_{\mathbb{H}}(x,y) \leq Length_{\mathbb{H}}(\sigma) < d_{\mathbb{H}}(\gamma(x),\gamma(y))$, because if no such path existed, $d_{\mathbb{H}}(\gamma(x),\gamma(y))$ would be a number greater than $d_{\mathbb{H}}(x,y)$ which is still a lower bound of $\{Length_{\mathbb{H}}(\sigma) | \sigma \text{is a piece wise continuously differentiable path from x to y}\}$. Fix σ. Observe that $\gamma(\sigma)$ is a piece wise continuously differentiable path from $\gamma(x)$ to $\gamma(y)$.

Next, observe the following:

(4)

$$\begin{aligned}
Length_{\mathbb{H}}(\gamma(\sigma(t)) &= \int_0^1 \frac{1}{\Im(\gamma(\sigma(t))}|(\gamma \circ \sigma)'(t)|dt \\
&= \int_0^1 \frac{2i}{\gamma(\sigma(t)) - \overline{\gamma(\sigma(t))}}|\gamma'(\sigma(t))\sigma'(t)|dt \\
&= \int_0^1 \frac{2i}{\frac{a\sigma(t)+b}{c\sigma(t)+d} - \frac{a\overline{\sigma(t)}+b}{c\overline{\sigma(t)}+d}}|\gamma'(\sigma(t))||\sigma'(t)|dt \\
&= \int_0^1 \frac{2i|c\sigma(t)+d|^2}{(a\sigma(t)+b)(c\overline{\sigma(t)}+d) - (c\sigma(t)+d)(a\overline{\sigma(t)}+b)}|\frac{ad-bc}{(c\sigma(t)+d)^2}||\sigma'(t)|dt \\
&= \int_0^1 \frac{2i(ad-bc)}{ac|\sigma(t)|^2 + bc\overline{\sigma(t)} + ad\sigma(t) + bd - ac|\sigma(t)|^2 - ad\overline{\sigma(t)} - bc\sigma(t) - bd}|\sigma'(t)|dt \\
&= \int_0^1 \frac{2i(ad-bc)}{ad(\sigma(t) - \overline{\sigma(t)}) - bc(\sigma(t) - \overline{\sigma(t)})}|\sigma'(t)|dt \\
&= \int_0^1 \frac{2i}{\sigma(t) - \overline{\sigma'(t)}}|\sigma'(t)|dt \\
&= \int_0^1 \frac{1}{\Im(\sigma(t))}|\sigma'(t)|dt = Length_{\mathbb{H}}(\sigma(t))
\end{aligned}$$

Thus, $Length_{\mathbb{H}}(\gamma(\sigma(t)) = Length_{\mathbb{H}}(\sigma(t))$. This implies that $Length_{\mathbb{H}}(\gamma(\sigma(t)) < d_{\mathbb{H}}(\gamma(x), \gamma(y))$, which is impossible considering the way $d_{\mathbb{H}}(\gamma(x), \gamma(y))$ is defined. Thus, by contradiction, $d_{\mathbb{H}}(x, y) = d_{\mathbb{H}}(\gamma(x), \gamma(y))$.

□

An important corollary of this result is the fact that Möbius transforms preserve geodesics.

Corollary 6.4.1. *If $\gamma \in M\ddot{o}b(\mathbb{H})$ and σ is a geodesic from a to b, then $\gamma(\sigma(t))$ is a geodesic from $\gamma(a)$ to $\gamma(b)$.*

Proof. Assume $\gamma(\sigma(t))$ is not a geodesic from $\gamma(a)$ to $\gamma(b)$. Then, there exists a path from $\gamma(a)$ to $\gamma(b)$, let us call it $\phi(t)$, which is shorter than $\gamma(\sigma(t))$. Since $\gamma^{-1}(t)$ is a Möbius transform, and Möbius transforms preserve length, it follows that the length of $\gamma^{-1}(\phi(t))$ is less than $\sigma(t)$. however, since $\gamma^{-1}(phi(t))$ is a path from a to b, this violates the assumption that $\sigma(t)$ is a geodesic. Thus, $\phi(t)$ cannot exist. □

The above theorem shows that Möbius transforms are isometries of $\mathbb{H}$. However, distance is not the only geometric structure which Möbius transforms preserve. In fact, Möbius transforms always send circles with real center and vertical lines to circles with real center and vertical lines.

Theorem 6.5. *All circles with real center and vertical lines can be represented in the form*

$$\alpha z\overline{z} + \beta z + \beta\overline{z} + \gamma = 0$$

for $\alpha, \beta, \gamma \in \mathbb{R}$.

Proof. The equation of a vertical line in the complex plane is $\Re(z) = k$, where $\Re(z)$ is the real component of z and $k \in \mathbb{R}$. $\Re(z) = \frac{1}{2}(z + \overline{z})$, which implies that $z + \overline{z} - 2k$. Letting $\alpha = 0$, $\beta = 1$, and $\gamma = -2k$, we see that the equation for a vertical line can be written in the form $\alpha z\overline{z} + \beta z + \beta\overline{z} + \gamma = 0$

Next, consider the equation of a circle in the complex plane with real center. Such a circle has the form $|z - a|^2 = r^2$, where $a, r \in \mathbb{R}$. Observe that

$$|z - a|^2 = (z - a)\overline{(z - a)} = (z - a)(\overline{z} - a) = z\overline{z} - az - a\overline{z} + a^2 = r^2.$$

And thus, letting $\alpha = 1$, $\beta = -a$, and $\gamma = a^2 - r^2$, we see that the equation for a circle with a real center can be written in the form

$$\alpha z\overline{z} + \beta z + \beta\overline{z} + \gamma = 0.$$

□

We will denote the set of vertical lines and circles with real center with $\mathcal{H}$.

Using this equation, we will show that under a Möbius transforms map elements of $\mathcal{H}$ to elements of $\mathcal{H}$

Lemma 6.6. *Möbius transforms map elements of $\mathcal{H}$ to elements of $\mathcal{H}$.*

Proof. Let $K \in \mathcal{H}$ and $\phi \in M\ddot{o}b(\mathbb{H})$. Then, by Theorem 6.5, K is the set of points in $\mathbb{C} \cup \{\infty\}$ such that they satisfy the equation

$$\alpha z\overline{z} + \beta z + \beta\overline{z} + \gamma$$

where $\alpha, \beta, \gamma \in \mathbb{R}$. Thus, the image of K is the set of all points $z \in \mathbb{C} \cup \{\infty\}$ such that

$$\alpha\phi^{-1}(z)\overline{\phi^{-1}(z)} + \beta\phi^{-1}(z) + \beta\overline{\phi^{-1}(z)} + \gamma = 0.$$

From Theorem 6.1 we see that $\phi^{-1}(z) = \frac{dz-b}{-cz+a}$ Thus, the equation becomes

$$\begin{aligned} 0 &= \alpha\frac{dz-b}{-cz+a}\overline{(\frac{dz-b}{-cz+a})} + \beta\frac{dz-b}{-cz+a} + \beta\overline{(\frac{dz-b}{-cz+a})} + \gamma \\ &= \alpha\frac{(dz-b)(d\overline{z}-b)}{(cz-a)(c\overline{z}-a)} + \beta(\frac{(dz-b)(-c\overline{z}+a) + (d\overline{z}-b)(-cz+a)}{|-cz+a|^2}) + \gamma \end{aligned} \tag{5}$$

Which simplifies to the equation

$$(\alpha d^2 - 2\beta cd + \gamma c^2)z\overline{z} + (-\alpha bd + \beta ad + \beta bc - \gamma ac)(z + \overline{z}) + (\alpha b^2 - 2\beta ab + \gamma a^2) = 0.$$

Letting $\alpha' = \alpha d^2 - 2\beta cd + \gamma c^2)$, $\beta' = -\alpha bd + \beta ad + \beta bc - \gamma ac$ and $\gamma' = \alpha b^2 - 2\beta ab + \gamma a^2$ we see that the equation takes the form $\alpha' z\overline{z} + \beta' z + \beta'\overline{z} + \gamma'$ with $\alpha', \beta', \gamma' \in \mathbb{R}$. Thus, the image of K under the Möbius transform is also in $\mathcal{H}$ □

The above theorem shows that the image of a vertical line or real circle under a Möbius transforms is still a vertical line or a real circle. This theorem will prove useful for proving an additional very useful theorem

Theorem 6.7. *Let $A, B \in \mathcal{H}$. Then, there exists a Möbius transform such which maps A to B.*

Proof. Note that by Theorem 6.2 the composition of two Möbius transforms is still a Möbius transform, so if we show that a series of Möbius transforms can turn any vertical line or circle with real center into another vertical line or circle with real center, that implies there exists a Möbius transform which does the same thing as the series of Möbius transforms.

We shall show that any element of $\mathcal{H}$ can be mapped to the imaginary axis via a series of Möbius transforms. Let $A \in \mathcal{H}$.

First, if A is a vertical line, simply translate the line horizontally until it is the imaginary axis, as horizontal translations are Möbius transforms.

If A is not a vertical line, then A is a circle with a real center. Say the circle has center a and radius r. Applying first the Möbius transform $f(z) = z - a$ and $g(z) = \frac{z}{r}$ shows that there exists a Möbius transform which sends A to the circle of unit radius and center 0.

Then, observe the behavior of the Möbius transform $h(z) = \frac{z-1}{z+1}$ on the circle of unit radius and center 0. The points 1 and i lie on the circle of unit radius and center 1.

$$h(i) = \frac{-1+i}{1+i} = \frac{-(1-i)(1-i)}{(1+i)(1-i)} = \frac{-(1-2i-1)}{2} = \frac{2i}{2} = i$$

and

$$h(1) = \frac{1-1}{1+1} = \frac{0}{2} = 0.$$

From Lemma 6.6, we know that the image of the unit circle under $h(z)$, which we will denote K, must be either a vertical line or a circle with real radius. Note that because $0 \in K$ and $i \in K$, K cannot be a circle with real radius, as the points which are equidistant from 0 and i, and which thus could possibly be the center for a circle containing 0 or i lie on the line $\Im(z) = 1/2$. Thus, K is a vertical line. Since $i \in K$, K must be the imaginary axis.

We have thus shown that for any $A \in \mathcal{H}$, there exists a Möbius transform which sends it to the imaginary axis. Let $A, B \in \mathcal{H}$. Then, let $\gamma_1, \gamma_2 \in Möb(\mathbb{H})$ be the Möbius transforms which sends A and B to the imaginary axis respectively. Then, $\gamma_2(\gamma_1^{-1}(z))$ is a Möbius transform by Theorem 6.2 and Theorem 6.1 and maps A to B. Since A and B are arbitrary, we are done. □

7. Geodesics, Angles, Area, and Triangles

If you will recall, previously in this paper the explicit form of the metric was stated, but it was not derived from the path-length definition. In this section, we will derive the metric, as well as state the definitions for Angles, Areas, and Triangles.

First, we shall describe geodesics and distances for points on the imaginary axis

Lemma 7.1. *For ai and bi such that $a, b \in \mathbb{R}$, $a, b > 0$, and $a < b$, then, the geodesic between these two points is a vertical line, and $d_{\mathbb{H}}(ai, bi) = \ln \frac{b}{a}$*

Proof. Suppose $\gamma(t) = x(t) + y(t)i$ and is a path from ai to bi. Then,

$$\begin{aligned} & Length_{\mathbb{H}}(\gamma(t)) \\ & \quad = \int_0^1 \frac{\sqrt{(x'(t))^2 + (y'(t))^2}}{y(t)} dt \\ & \quad \geq \int_0^1 \frac{|y'(t)|}{y(t)} dt \\ & \quad \geq \int_0^1 \frac{y'(t)}{y(t)} dt \\ & \quad = \ln(y(t))\Big|_0^1 \\ & \quad = \ln(\frac{b}{a}) \end{aligned} \tag{6}$$

This shows that $Length_{\mathbb{H}}(\gamma(t)) \geq \ln(\frac{b}{a})$. Next, we need to show that a vertical line segment between ai and bi has Let $\sigma(t) = (b-a)ti + ai$, which is a vertical line segment from ai to bi. Then,

$$Length_{\mathbb{H}}(\sigma(t)) = \int_0^1 \frac{(b-a)}{(b-a)t + a} dt = \ln(t + \frac{a}{b-a})\Big|_0^1 = \ln(\frac{b}{a}).$$

This implies that $Length_{\mathbb{H}}(\gamma(t)) \geq Length_{\mathbb{H}}(\sigma(t))$ for all paths $\gamma(t)$ from ia to ib. Thus, $d_{\mathbb{H}}(ai, bi) = \ln(\frac{b}{a})$, and $\sigma(t)$ which is a vertical line, is a geodesic. ☐

From this result, we can then prove that all geodesics are either vertical lines or circles of real radius

Theorem 7.2. *The unique geodesic between any two points in* $\mathbb{H}$ *is the unique circle of real radius or vertical line which passes through those two points*

Proof. By Theorem 6.7, there is a Möbius transform which maps the imaginary Axis to the vertical line or circle of real radius which contains the two points. By Corollary 6.4.1, since the imaginary axis is a geodesic, so is the vertical line or circle of real radius which passes through the two points ☐

Now that we have described the geodesics, defining the explicit form of the metric is a matter of integrating along the geodesics. However, there is an easier way to determine the metric, which will be described in the following proof.

Theorem 7.3. *The explicit form of the metric in* $\mathbb{H}$ *is*

$$d_{\mathbb{H}}(z, w) = cosh^{-1}(1 + \frac{|z-w|^2}{2\Im(z)\Im(w)}).$$

Proof. Let $z, w \in \mathbb{H}$. Then, Define $LHS(z, w) = cosh(d_{\mathbb{H}(z,w)})$ and $RHS(z, w) = 1 + \frac{|z-w|^2}{2\Im(z)\Im(w)}$. Let γ be the Möbius transform which sends z and w to the imaginary axis. Let $\gamma(z) = ai$ and $\gamma(w) = bi$ for real numbers a and b. Then, we shall show that

$$LHS(z, w) = LHS(\gamma(z), \gamma(w)) = RHS(\gamma(z), \gamma(w)) = RHS(z, w).$$

Firstly, the fact that $LHS(z, w) = LHS(\gamma(z), \gamma(w))$ follows from the fact that γ is an isometry

Next, note that

$$LHS(ai, bi) = \cosh(\ln(\frac{b}{a})) = \frac{\frac{b}{a} + \frac{a}{b}}{2} = \frac{a^2 + b^2}{2ab}.$$

Also,

$$RHS(ai, bi) = 1 + \frac{|a - b|^2}{2ab} = 1 + \frac{a^2 - 2ab + b^2}{2ab} = \frac{a^2 + b^2}{2ab}.$$

Thus, $LHS(\gamma(z), \gamma(w)) = RHS(\gamma(z), \gamma(w))$

Finally, let $\gamma(z) = \frac{az+b}{cz+d}$. Then,

$$\begin{aligned}
RHS(\gamma(z), \gamma(w)) & \\
&= 1 + \frac{|\gamma(z) - \gamma(w)|^2}{2\Im(\gamma(z)\Im(\gamma(w))} \\
&= 1 + \frac{|\frac{az+b}{cz+d} - \frac{aw+b}{cw+d}|^2}{2\frac{(ad-bc)^2}{|cz+d|^2|cw+d|^2}\Im(z)\Im(w)} \\
&= \frac{|cz+d|^2|cw+d|^2|\frac{(aczw+bcw+adz+bd-aczw-bcz-adw-bc)}{|cz+d||cw+d|}|^2}{2(ad-bc)^2\Im(z)\Im(w)} \\
&= \frac{|(ad-bc)(z-w))|^2}{2(ad-bc)^2\Im(z)\Im(w)} \\
&= \frac{|(z-w))|^2}{2\Im(z)\Im(w)} \\
&= RHS(z, w)
\end{aligned} \tag{7}$$

This implies that $LHS(z, w) = RHS(z, w)$, and thus that $d_{\mathbb{H}} = \cosh^{-1}(\frac{|z-w|^2}{2\Im(z)\Im(w)})$ □

Before we continue, we must first define the inner product, norm, angle and area in Hyperbolic Space.

Definition 9. *The inner product of two vectors $\vec{v}$ with components v_1 and v_2 and $\vec{w}$ with components w_1 and w_2 at the position z in Hyperbolic space is*

$$\langle \vec{v}, \vec{w} \rangle := \frac{v_1 w_1 + v_2 w_2}{\Im(z)^2}.$$

The norm of a vector at z in $\mathbb{H}$ is defined analogously to Euclidean Space, except with the Hyperbolic inner product rather than the Euclidean One.

Definition 10. $||\vec{v}|| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$

Definition 11. *Let γ and σ be two paths, and let $\gamma(0) = \sigma(0)$, that is, the paths intersect. Then, the angle between the two paths at the point $\gamma(0) = \sigma(0)$, denoted θ, is defined as*

$$\theta = \cos^{-1}\Big(\frac{\langle \gamma'(0), \sigma'(0) \rangle}{||\gamma'(0)||\sigma'(0)||}\Big),$$

where $\langle \cdot, \cdot \rangle$ is the inner product.

Note that although the definition of the norm and inner product in $\mathbb{H}$ is different from the Euclidean definition, because the inner product is scaled by a factor of $\frac{1}{\Im(z)^2}$ and the norm is scaled by a factor of $\frac{1}{\Im(z)}$, the two effects cancel, and the notion of angle in Hyperbolic Space is the exact same as Euclidean Space.

Definition 12. *Let A be some subset of $\mathbb{H}$. Then, the area of A is*

$$\int\int_A \frac{1}{\Im(z)^2}dz.$$

As one would expect, the notions of angle and area in $\mathbb{H}$ are invariant under Möbius transform.

Theorem 7.4. *Angle is invariant under Möbius transforms.*

Proof. Let $\gamma \in M\ddot{o}b(\mathbb{H})$. Then, let $\gamma(x+yi) = u(x,y)+v(x,y)i$. By a theorem in from complex analysis, $u_x = v_y$ and $u_y = -v_x$ The tangent vector $\vec{v}$ after applying the transform γ is $D\vec{v}$, where D is the matrix

$$\begin{pmatrix} u_x & u_y \\ -u_y & u_x \end{pmatrix}.$$

Observe that

$$D^T D = \begin{pmatrix} u_x^2+u_y^2 & 0 \\ 0 & u_x^2+u_y^2 \end{pmatrix}.$$

This implies that $\frac{D}{\sqrt{u_x^2+u_y^2}}$ is an orthogonal matrix. Since $D = \frac{D}{\sqrt{u_x^2+u_y^2}}\sqrt{u_x^2+u_y^2}$, this implies D is the composition of an orthogonal matrix and multiplication by a constant. Since both these operations preserve the euclidean angles between vectors, and euclidean and hyperbolic angles are equivalent, γ preserves the angles between vectors. □

Theorem 7.5. *Area is invariant under Möbius Transform*

Proof. Let $A \subset \mathbb{H}$. Let $\gamma = \frac{az+b}{cz+d} \in M\ddot{o}b(\mathbb{H})$. Then, the determinant of the Jacobian matrix of the transform is

$$\frac{(ad-bc)^2}{((cx+d)^2+c^2y^2)^2}.$$

In addition, let $h(x,y) = \frac{1}{y^2}$. Then,

$$(h\circ\gamma(x,y)) = (\frac{(cx+d)^2+c^2y^2}{(ad-bc)y})^2.$$

From this, we conclude that the area of $\gamma(A)$ is

$$\begin{aligned}
&\int\int_{\gamma(A)} \frac{1}{y^2}dxdy \\
&= \int\int_A h\circ\gamma(z)\frac{(ad-bc)^2}{((cx+d)^2+c^2y^2)^2}dxdy \\
&= \int\int_A (\frac{(cx+d)^2+c^2y^2}{(ad-bc)y})^2\frac{(ad-bc)^2}{((cx+d)^2+c^2y^2)^2}dxdy \qquad (8) \\
&= \int\int_A \frac{1}{y^2}dxdy
\end{aligned}$$

Thus, area is preserved under Möbius transform. □

Finally, we define two more concepts, the definition of the boundary of $\mathbb{H}$, which is in turn necessary to define the generalized notion of a triangle in $\mathbb{H}$

Definition 13. *We define the boundary of $\mathbb{H}$, called $\partial\mathbb{H}$ to be the set $\mathbb{R} \cup \{\infty\}$*

Definition 14. *We define a triangle in $\mathbb{H}$ as follows. For any 3 points in $\mathbb{H}$ or $\partial\mathbb{H}$, we define a triangle to be the set bounded by the geodesics connecting these 3 points.*

Note that because we allows some of these points to be in $\partial\mathbb{H}$, in some triangles, the geodesics may not meet in $\mathbb{H}$

8. The Gauss - Bonnet Theorem

Now, all that remains is to prove the Gauss - Bonnet Theorem. This Theorem demonstrates a particularly interesting fact about hyperbolic geometry, that is, the area of a shape is related to it's internal angles. In Euclidean Geometry, the area of a triangle is independent of its angles. We can scale a triangle up or down without affecting it's angles. However, this is not true in Hyperbolic Geometry, as we will soon prove.

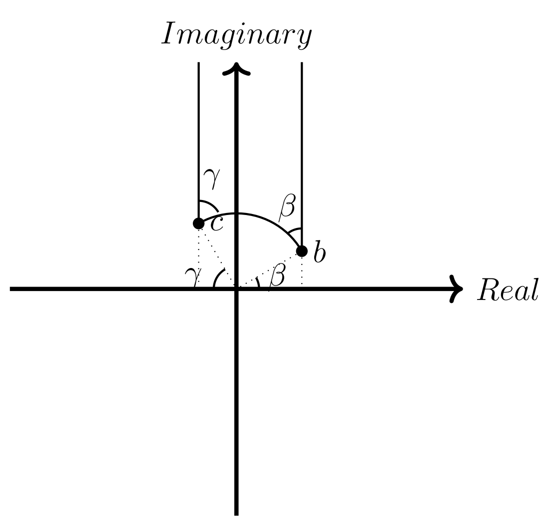
Theorem 8.1. *Let $\triangle$ be a triangle in $\mathbb{H}$. Then, the area of $\triangle$, denoted Area($\triangle$), is $\pi - (\alpha + \beta + \gamma)$, where α, β, γ, are the internal angles of the triangle.*

Proof. First, we prove this theorem for the special case where one vertex of the triangle is in $\partial\mathbb{H}$.

Recall that Möbius transforms preserve angle and area. Thus, if the theorem holds for a triangle after a certain set of Möbius transforms, it will hold for the triangle before those Möbius transforms.

Let the vertexes of the triangle be a, b, c, with $a \in \partial\mathbb{H}$.

Firstly, if the vertex of the triangle in $\partial\mathbb{H}$ is not ∞ apply the transform $\frac{z-(a+1)}{z-a}$, which sends a to ∞. Next, apply a horizontal transformation to send the center of the geodesic connecting b and c to the origin. Finally, apply the transform kz to scale the geodesic from b to c to radius 1. The resulting triangle looks like this



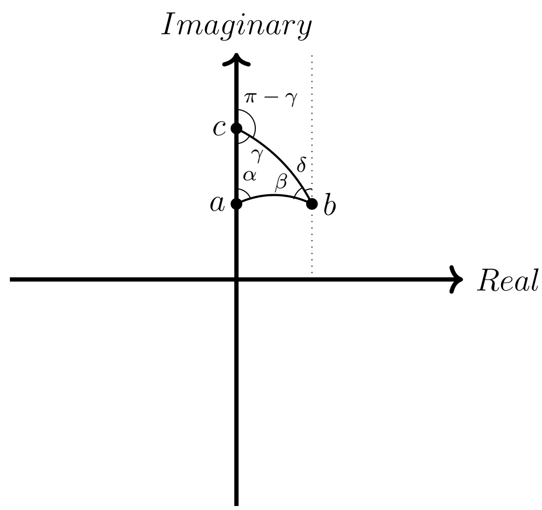
We can calculate the area of this triangle by integrating.

$$
\begin{aligned}
&Area(\triangle)\\
&= \int_{\Re(c)}^{\Re(b)} \int_{\sqrt{1-x^2}}^{\infty} \frac{1}{y^2} dy dx\\
&= \int_{\Re(c)}^{\Re(b)} \frac{1}{\sqrt{1-x^2}} dx \qquad (9)\\
&= \int_{\pi-\gamma}^{\beta} -1 d\theta \text{ (After substituting } x = cos(\theta))\\
&= \pi - \beta - \gamma
\end{aligned}
$$

This proves the theorem in the case where $\alpha = 0$ (which happens when $a \in \partial\mathbb{H}$)

Next, we will use this result to prove the theorem for the general triangle.

Let the points a, b, c be the vertices of a triangle, and α, β, γ, be the respective internal angles. By Theorem 6.7, there exists a Möbius Transform which sends a and c to the imaginary axis. Then, let δ be the angle between b and the vertical line passing through b after this transform.



By the previous result, we know $Area(ab\infty) = \pi - (\beta + \delta) - \alpha$, and $Area(cb\infty) = \pi - (\pi - \gamma) - \delta = \gamma - \delta$

Thus, since $Area(abc) = Area(ab\infty) - Area(cb\infty) = \pi - (\beta + \delta) - \alpha - \gamma + \delta = \pi - \alpha - \beta - \delta$

This proves the theorem.

□

The Gauss-Bonnet theorem has many interesting consequences. For example, it implies that triangles have a maximum area, when all 3 vertecies are on $\partial\mathbb{H}$, and thus all internal angles are 0. This shape is known as an "ideal triangle", and has area π. It also implies that the sum of angles in a Hyperbolic Triangle is always less than π.

9. Bibliography

References

[1] Charles Walkden (2019), Hyperbolic Geometry, Retrieved July 6th, 2023

[2] Michael Hvidsten (2012), Exploring Geometry, Retrieved July 6th, 2023