The Independence of Ext from Choice of Projective Resolution

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Euler Circle

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Background: Modules

A module is a set M which is an abelian group under "addition" together with a "scalar multiplication" $R \times M \to M$ that satisfies

$$a(bm) = (ab)m$$

$$a(m + m') = (am + am')$$

$$(a + b)m = am + bm$$

$$1m = m$$

for all $a, b \in R$, $m, m' \in M$.

Background: Modules

An R-module (with R a ring) is a set M which is an abelian group under "addition" together with a "scalar multiplication" $R \times M \to M$ that satisfies

Examples

1. R
2. $\mathbb{F}^n$, $\mathbb{F}$ a field

Background: Hom

For modules M, M', $\mathrm{Hom}(M, M')$ is the set of all homomorphisms $M \to M'$.

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$$f + f' := (f + f')(m) = f(m) + f'(m)$$

Cochains

For a fixed ring R, a cochain is a sequence of R-modules and R-module homomorphisms $\{C^\bullet, d^\bullet\}$ with $d^n\colon C^n \to C^{n+1}$ such that $d^i \circ d^{i+1} = 0$.

$$\cdots \xrightarrow{d^{n-2}} C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \xrightarrow{d^{n+1}} \cdots$$

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Essentially, $\operatorname{im} d^{i-1} \subseteq \ker d^i$.

If $\operatorname{im} d^{i-1} = \ker d^i$, then the sequence is said to be exact.
Example:

$$0 \longrightarrow A \xrightarrow{\iota} A \oplus B \xrightarrow{\pi_2} B \longrightarrow 0$$

Cohomology

Let $C^\bullet$ be a cochain complex.

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Example

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z}/8\mathbb{Z} \xrightarrow{\times 4} \mathbb{Z}/8\mathbb{Z} \xrightarrow{\times 4} \mathbb{Z}/8\mathbb{Z} \xrightarrow{\times 4} \cdots$$

With the first nonzero module at index 0, $H^n = 0$ for $n < 0$, $H^0 = \mathbb{Z}/4\mathbb{Z}$, and $H^n = \mathbb{Z}/2\mathbb{Z}$ for $n > 0$.

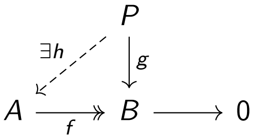
Projective Resolution

A projective resolution of A is an exact sequence

$$0 \longleftarrow A \xleftarrow{\epsilon'} P_0 \xleftarrow{d_0'} P_1 \xleftarrow{d_1'} \cdots$$

where each P_i is *projective.*

Projective Resolution

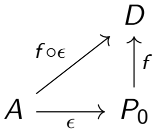


Ext

Let $\{P_A^\bullet\}$ be

$$0 \longrightarrow \mathrm{Hom}(A, D) \xrightarrow{\ \epsilon\ } \mathrm{Hom}(P_0, D) \xrightarrow{\ d_0\ } \mathrm{Hom}(P_1, D) \xrightarrow{\ d_1\ } \cdots$$

Where the maps d_i (and ϵ) are formed by precomposition: for $f \in \mathrm{Hom}(X, D)$, $d_i(f) \mapsto f \circ d_i'$, which reverses the direction of the arrows.



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$$\mathrm{Ext}_R^n(A, D) = H^n(P_A^\bullet)$$

for $n \neq 0$,

$$\mathrm{Ext}_R^0(A, D) = \ker d_0.$$

Independence

$$\begin{array}{ccccccccc}
0 & \longleftarrow & A & \longleftarrow & P_0 & \longleftarrow & P_1 & \longleftarrow & \cdots \\
 & & \Big\downarrow{\scriptstyle f} & & & & & & \\
0 & \longleftarrow & A' & \longleftarrow & P'_0 & \longleftarrow & P'_1 & \longleftarrow & \cdots
\end{array}$$

Independence

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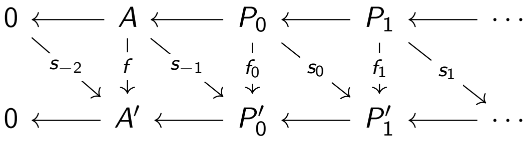
Independence



$$f_n = d'_{n+1} s_n + s_{n-1} d_n$$

Independence

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& \nwarrow^{s_{-2}} & \uparrow f & \nwarrow^{s_{-1}} & \uparrow f_0 & \nwarrow^{s_0} & \uparrow f_1 & \nwarrow^{s_1} & \\
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$\uparrow$ is still true!

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set $f = 0$

Independence

Now let $A = A'$, $f = id_A$.

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$$g'_n \circ f'_n = id_{H^n(P^\bullet_A)}$$

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\end{array}$$

$$g_n' \circ f_n' = id_{H^n(P_A^\bullet)}$$

And swapping the roles of A and A' yields $f_n' \circ g_n' = id_{H^n(P_A'^\bullet)}$, so for *any* projective resolution of A, the cohomology groups are isomorphic, and in this sense, Ext is independent of choice of projective resolution.