Introduction to Wilson's Oddness Theorem

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Euler Circle Presentations

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Game theory

Game theory is the study of mathematical models of strategic interactions among rational agents.
Essentially, we study a theoretical framework for conceiving social situations among competing players. Game theory can be seen as a science for strategy.

Definition

The equilibrium points are the situations in which a player will continue with their chosen strategy, having no incentive to deviate from it, after taking into consideration the opponent's strategy

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The Utility function U_i shows the payoff of a given strategy.
Payoff of pure strategy n-tuple $a^m : U_i(a^m) = u_i^m$
Payoff of mixed strategy n-tuple $p(p_1, \ldots, p_n)$:
$U_i(p) = \sum_{m=1}^{K} [\Pi_{i=1}^{n} q_i^m(p_i)] u_i^m$

Equilibrium Points

A $n-$tuple S is an equilibrium point if and only if for every i:

$$U_i(S) = \max_{all r_i' s}[U_i(S; r_i)]$$

which by linearity of p_i gives us:

$$\max_{\alpha}[U_i(S; \pi_{i\alpha})] = \max_{all r_i' s}[U_i(S; r_i)]$$

. Define $p_{i\alpha}(S) = p_i(S; \pi_{i\alpha})$. Then we have S is an equilibrium point if:

$$U_i(S) = \max_{\alpha} U_{i\alpha}(S)$$

.

i indicates a player, α indicates a pure strategy of a player, r_i indicates a mixed strategy of i and π_{ia} indicates i's α^{th} pure strategy.

Prisoner's Dilemma

Let's look at a famous situation in game theory called the Prisoner's Dilemma.
A situation has two prisoners *a* and *b* and the following outcomes below are possible.

Prisoner A \ Prisoner B	Remain silent	Confess
Remain silent	A gets 2 years B gets 2 years	A gets 8 years B gets 1 year
Confess	A gets 1 year B gets 8 years	A gets 5 years B gets 5 years

This game has only one equilibrium point which is both *a* and *b* confessing.

Why do we study Game Theory?

We use game theory for optimal decision-making of independent and competing actors in a strategic setting. Using game theory, real-world scenarios for such situations as pricing competition and product releases (and many more) can be laid out and their outcomes predicted. It can help assess how efficiently communities make decisions when there is no cooperation between players in a game.
Essentially, game theory allows us to maximise payoff of a situation using equilibrium points, which is usually the aim of game theory.

Wilson's Oddness Theorem

Wilson's Oddness Theorem states that:

Theorem

In almost all finite games, the number of equilibrium points is finite and odd

We will look at the *almost all* nature of this theorem in the upcoming slides.

What is a Carrier?

For any strategy n-tuple $p = (p_1, \ldots, p_n)$ the carrier of $C(p)$ will be defined as the union of carrier of its component strategies as:

$$C(p) = \bigcup_{i=1}^{n} C(p_i).$$

Additional Explanation: The carrier of p_i is the set $C(p_i)$ of all pure strategies a_i^k to which the mixed strategy p_i assigns positive probabilities $p_i^k > 0$. If this carrier contains only one pure strategy a_i^k, then $a_i^k = p_i$. However, if $C(p_i)$ contains all K_i pure strategies of player i, then p_i will be called a *complete mixed* strategy. And lastly, if p_i doesn't have only one or all pure strategies, it will be called an *incomplete mixed* strategy.

What does a Quasi-Strong Game mean?

A quasi strong game is a game that has *only* quasi-strong equilibrium points.

Definition

Quasi Strong Equilibrium Point: p is quasi strong if no player i has pure-strategy best replies to $\overline{p}_i$ other than the pure strategies belonging to carrier $C(p_i)$ of his equilibrium point p_i.

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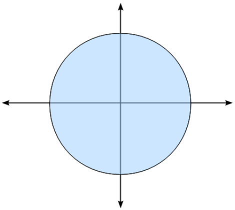
These proofs rely heavily on topological properties of the solution graph T wherein T is the locus of all solutions (t, p) to the simultaneous equations and inequalities.

'Almost All'

Let's think about what the 'Almost All' means in Wilson's Oddness Theorem. It's really unusual to have something like this in math.

Suppose we have a dartboard of area 1. A dart that will always land on the dartboard will have a probability of 1 of landing on the entire dartboard. What happens if you pick a region that is a single point, say the origin? As the radius gets smaller and smaller and closer and closer to 0, the area becomes infinitesimally small. The limiting value is when the disk "turns into" a single point, and this will have an area of 0.



An Exceptional Case

Consider the following game, player x and player y get 1 a piece if they play (Up, Left) and they get nothing otherwise.

As shown on the right, Left-Up is one equilibrium point and Right-Down is one equilibrium point. That gives 2 (an even number) of equilibrium points!
In fact, in the Up-Left box, if we change 1 to any positive real number, there will always exist 2 equilibrium points.

Player 'y'

Player 'x'		Left	Right
	Up	1, 1	0, 0
	Down	0, 0	0, 0