Hardy-Littlewood circle method

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Overview

- Outline of Hardy-Littlewood circle method

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- Waring's problem

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- Waring's problem
- Summary of application to Waring's problem

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- Evaluate the value of the integral over major arcs.
- Bound the minor arc.
- Gather information about the problem.

Waring's Problem

For $k \in \mathbb{N}$ with $k \geq 2$, let $G(k)$ denote the least integer $s = s(k)$ such that for all $n \in \mathbb{N}$ *sufficiently large*, there exist $x_1, ..., x_s \in \mathbb{N}$ such that

$$n = x_1^k + x_2^k + \cdots + x_s^k?$$

Getting started

Fix $k, s \in \mathbb{N}$ with $k \geq 2$. For $n \in \mathbb{N}$ define

$$R_{s,k}(n) = \#\{x_1, ..., x_s \in \mathbb{N} : n = x_1^k + \cdots + x_s^k\}.$$

Additionally, for $\alpha \in \mathbb{R}$, let $e(\alpha) = e^{2\pi i \alpha}$. Then, for $m \in \mathbb{Z}$ we have

$$\int_0^1 e(\alpha m) d\alpha = \begin{cases} 1, & \text{if } m = 0, \\ 0, & \text{otherwise.} \end{cases}$$

An Integral Representation

Using that,

$$\int_0^1 e(\alpha(x_1^k + \cdots + x_s^k - n))d\alpha = \begin{cases} 1, & \text{if } n = x_1^k + \cdots + x_s^k, \\ 0, & \text{if } n \neq x_1^k + \cdots + x_s^k. \end{cases}$$

Additionally, note that $x_i \leq \lfloor n^{1/k} \rfloor$. Let $N = \lfloor n^{1/k} \rfloor$, so $1 \leq x_i \leq N$.

A New Representation

With that, the problem can be represented as

$$\sum_{1 \le x_1 \le N} \cdots \sum_{1 \le x_s \le N} \int_0^1 e(\alpha(x_1^k + \cdots + x_s^k - n))d\alpha = R(n).$$

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$$= \int_0^1 \left(\sum_{1\leq x_1\leq N} e(\alpha x_1^k) \right) \cdots \left(\sum_{1\leq x_s\leq N} e(\alpha x_s^k) \right) e(-\alpha n) d\alpha$$

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$$=\int_0^1 f^s(\alpha)e(-\alpha n)d\alpha$$

where

$$f(\alpha)=\sum_{x=1}^{N} e(\alpha x^k).$$

Major and Minor Arcs

The idea is now to write

$$R(n) = \int_{\mathfrak{M}} f^s(\alpha) e(-\alpha n) d\alpha + \int_{\mathfrak{m}} f^s(\alpha) e(-\alpha n) d\alpha,$$

where $\mathfrak{M}$ and $\mathfrak{m}$ are disjoint and $\mathfrak{M} \cup \mathfrak{m}$ is a unit interval.

Major Arcs

The goal is to show that

$$\int_{\mathfrak{M}} f^s(\alpha)e(-\alpha n)d\alpha \gg n^{s/k-1}.$$

The major arcs $\mathfrak{M}(q,a) = \{\alpha \in \mathbb{R} : |\alpha - a/q| \leq N^{\nu-k}\}$ where $a, q \in \mathbb{N}$, $1 \leq a \leq q \leq N^\nu$, $(a,q) = 1$, and ν is a sufficiently small positive number. Now,

$$\mathfrak{M} = \bigcup_{q \leq N^\nu} \bigcup_{a=1}^{q} \mathfrak{M}(q,a).$$

Minor Arcs

The goal is to show that

$$\int_{\mathfrak{m}} f^s(\alpha)e(-\alpha n)d\alpha = o(n^{s/k-1}).$$

The minor arcs $\mathfrak{m}$ are

$$\mathfrak{m} = [0,1)\backslash\mathfrak{M}.$$

An important theorem

Theorem

Suppose that $\alpha \in \mathbb{R}$. Then for every real number $X \geq 1$, there exist $a, q \in \mathbb{Z}$ satisfying $(a, q) = 1$ and $1 \leq q \leq X$ such that

$$\left| \alpha - \frac{a}{q} \right| \leq \frac{1}{q^X}.$$

Proof of an important theorem

Consider first the $\lfloor X \rfloor$ numbers

$$\{t\alpha\}, \qquad t = 1, .., \lfloor X \rfloor,$$

and the $\lfloor X \rfloor + 1$ intervals

$$I_j = \left[\frac{j-1}{\lfloor X \rfloor + 1}, \frac{j}{\lfloor X \rfloor + 1} \right) \qquad j = 1, ..., \lfloor X \rfloor + 1.$$

If one of the $\lfloor X \rfloor$ numbers lies in I_1 or $I_{\lfloor X \rfloor + 1}$ then the theorem holds with $q = t$.

Proof of an important theorem (cont.)

Otherwise, by the pigeonhole principle one of the remaining intervals must contain two of the $\lfloor X \rfloor$ numbers. This means there exist integers t_1, t_2 satisfying $1 \leq t_1 < t_2 \leq \lfloor X \rfloor$ and an integer $i = 2, .., \lfloor X \rfloor$ such that $\{t_1\alpha\}$, $\{t_2\alpha\} \in I_i$ such that

$$|\{t_2\alpha\} - \{t_1\alpha\}| \leq \frac{1}{\lfloor X \rfloor + 1} \leq \frac{1}{X}.$$

Then taking $q = t_2 - t_1$ and $a = [t_2\alpha] - [t_1\alpha]$ completes it.

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The bound has been improved many times.

In Conclusion

Thank you.