HARDY-LITTLEWOOD CIRCLE METHOD

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Abstract. The circle method was first used by Hardy and Ramanujan in a paper on the partition function. The goal is to approximate the coefficients of a power series. Originally this was with the use of contour integrals near the unit circle. As a result, it is called the circle method. The goal of this paper is to examine the Hardy-Littlewood circle method as it is applied to Waring's problem.

1. Introduction

The circle method first was used in a paper by Hardy and Ramanujan about partitions. It was further developed by Hardy and Littlewood, and as a result it is referred to as the Hardy- Littlewood circle method. The method has been modified for use to study numerous problems in additive number theory. An important advancement of the method is due to Vinogradov. It replaces the infinite power series originally used by a finite exponential sum.

An early application of the method was on Waring's problem. In 1770, Edward Waring conjectured that "Every integer is a cube or the sum of two, three, ... nine cubes; every integer is also the square of a square, or the sum of up to nineteen such; and so forth." [VW02] This led to the following:

Question 1.1. *For $k \in \mathbb{N}$ with $k \geq 2$, what is the least integer $s = s(k)$ such that for all $n \in \mathbb{N}$, there exist $\boldsymbol{x} = x_1, \ldots, x_s \in \mathbb{N} \cup \{0\}$ such that*

$$n = x_1^k + \cdots + x_s^k?$$

Let $g(k)$ denote the least s such that the above equation holds for all $n \in \mathbb{N}$.

In 1770, Lagrange proved that $g(2) = 4$. The question was then altered somewhat to be of more interest.

Question 1.2. *For $k \in \mathbb{N}$ with $k \geq 2$, what is the least integer $s = s(k)$ such that for all $n \in \mathbb{N}$ sufficiently large, there exist $x_1, \ldots, x_s \in \mathbb{Z}^+$ such that*

$$n = x_1^k + \cdots + x_s^k?$$

Let $G(k)$ denote the lest s such that the above equation holds for all $n \in \mathbb{N}$.

It is the statement in Question 1.2 that leads to the following theorem.

Theorem 1.3. *Suppose that $k \geq 2$ is a fixed integer. Suppose additionally that $s \geq 2^k + 1$. Then there exists an M such the for every $n \in \mathbb{N}$ with $n > M$, and there exist non-negative integers $x_1, \ldots, x_s$ such that $n = x_1^k + \cdots + x_s^k$.*

Notably this is not the best possible. See [VW02] for additional bounds.

Date: July 16, 2023.

2. The Circle Method Overview

The application of the circle method is as follows:

- Construct a generating function that represents the problem.
- From this generating function, find an integral representation of the function.
- Separate this integral into the integral over the major $\mathfrak{M}$ and minor $\mathfrak{m}$ arcs,

$$\int_0^1 f(x)dx = \int_{\mathfrak{M}} f(x)dx + \int_{\mathfrak{m}} f(x)dx$$
$$= \text{main term} + \text{error term}.$$

- Evaluate the value of the integral over major arcs.
- Bound the minor arc.
- Gather information about the problem.

The goal will be to apply these steps to Waring's problem.

3. Acknowledgements

The author would like to thank Simon Rubinstein-Salzedo and Kishan Jani for their input and insights on this paper. Additionally, thank you to my grandfather for lending his likeness to the proof duck .

4. Background

For clarity, the following notation will be defined as follows:

- (a,b) = gcd(a,b).
  Example: Numbers $a, b \in \mathbb{Z}$ with $(a, b) = 1$ are coprime.
- $\lfloor x \rfloor$ is defined as the largest integer y such that $y \leq x$.
  Example: $\lfloor 8.67 \rfloor = 8$.
- $\{x\} = x - \lfloor x \rfloor$.
  Example: $8.67 - \lfloor 8.67 \rfloor = \{.67\}$.
- The notation $f(x) = O(g(X))$ means that there exists a $C > 0$ and some x_0 such that $|f(x)| \leq Cg(x) \forall x \geq x_0$ which will be denoted here as $f(x) \ll g(x)$ or $g(x) \gg f(x)$. Example: $4x^2 + 9x \ll x^2$.
- The notation $f(x) = o(g(x))$ means that $\lim_{x\to\infty} \frac{f(x)}{g(x)} \to 0$. In other words, $f(x)$ is of smaller order than $g(x)$.
  Example: $9x = o(x^2)$.
- If $f(x) = g(x) + o(g(x))$, the value of $f(x)$ depends on the first term asymptotically because the second term is negligible.
  Example: $4x^2 + 9x \sim 4x^2$.

5. Initial Steps

Firstly, suppose $m \in \mathbb{Z}$. Then,

$$\int_0^1 e(m\alpha)d\alpha$$
$$= \begin{cases} 1, & \text{if } m = 0 \\ 0, & \text{if } m \neq 0 \end{cases}.$$

Considering $R(n) = \#\{x_1, \ldots, x_s \in \mathbb{N}^s : n = x_1^k + \cdots x_s^k\}$, and using the above integral, if $n, x_1^k, \ldots, x_s^k \in \mathbb{N}$, then

$$\int_0^1 e(\alpha(x_1^k + \cdots + x_s^k - n))d\alpha$$
$$= \begin{cases} 1, & \text{if } n = x_1^k + \cdots + x_s^k \\ 0, & \text{if } n \neq x_1^k + \cdots + x_s^k \end{cases}$$

so that

$$R(n) = \sum_{x_1} \cdots \sum_{x_s} \int_0^1 e(\alpha(x_1^k + \cdots + x_s^k - n))d\alpha. \tag{5.1}$$

Additionally, since $n = x_1^k + \cdots x_s^k$, it can be observed that $x_i \leq n^{\frac{1}{k}}$ and thus

$$\begin{aligned} R(n) &= \sum_{1 \leq x_1 \leq n^{1/k}} \cdots \sum_{1 \leq x_s \leq n^{1/k}} \int_0^1 e(\alpha(x_1^k + \cdots + x_s^k - n))d\alpha \\ &= \int_0^1 \sum_{1 \leq x_1 \leq n^{1/k}} \cdots \sum_{1 \leq x_s \leq n^{1/k}} e(\alpha x_1^k) \cdots e(\alpha x_s^k) e(-\alpha n) d\alpha \\ &= \int_0^1 \left(\sum_{1 \leq x_1 \leq n^{1/k}} e(\alpha x_1^k) \right) \cdots \left(\sum_{1 \leq x_s \leq n^{1/k}} e(\alpha x_s^k) \right) e(-\alpha n) d\alpha \\ &= \int_0^1 f^s(\alpha) e(-\alpha n) d\alpha, \end{aligned} \tag{5.2}$$

where

$$f(\alpha) = \sum_1^N e(\alpha x^k)$$

with $N = [x^{1/k}]$. Then, one can write

$$R(n) = \int_{\mathfrak{m}} f^s(\alpha) e(-\alpha n) d\alpha + \int_{\mathfrak{M}} f^s(\alpha) e(-\alpha n) d\alpha. \tag{5.3}$$

Here, $\mathfrak{M}$ and $\mathfrak{m}$ are two disjoint sets with $\mathfrak{M} \cup \mathfrak{m}$ being a unit interval. For every $a, q \in \mathbb{N}$, with $1 \leq a \leq q \leq N^\nu$ and $(a, q) = 1$, let

$$\mathfrak{M}(q, a) = \{\alpha \in \mathbb{R} : |\alpha - a/q| \leq N^{v-k}.$$

These intervals $\mathfrak{M}(q, a)$ are called the major arcs. Thus,

$$\mathfrak{M} = \bigcup_{q \leq N^\nu} \bigcup_{a=1}^{q} \mathfrak{M}(q, a).$$

Let

$$U = (N^{\nu-k}, N^{\nu-k} + 1]$$

and

$$\mathfrak{m} = U \backslash \mathfrak{M}.$$

These $\mathfrak{m}$ are the minor arcs.

6. Minor Arcs

The goal here is to show that

$$\int_{\mathfrak{m}} f^s(\alpha)e(-\alpha n)d\alpha = o(n^{s/k-1}).$$

Firstly, consider that

$$\left|\int_{\mathfrak{m}} f^s(\alpha)e(-\alpha n)d\alpha\right| \leq \int_{\mathfrak{m}} |f^s(\alpha)|d\alpha \leq \left(\sup_{\alpha\in\mathfrak{m}} |f(\alpha)|\right)^{s-2^k} \int_0^1 |f(\alpha)|^{2^k} d\alpha. \tag{6.1}$$

The results will need the following three theorems. The proof of Theorem 6.1 and Theorem 6.3 will be later in the paper.

Theorem 6.1. *Suppose $a, q \in \mathbb{N}$ satisfy $(a, q) = 1$. Suppose as well that $\alpha \in \mathbb{R}$ satisfies $|\alpha - \frac{a}{q}| \leq q^{-2}$. Then*

$$|f(\alpha)| \ll N^{1+\epsilon}(q^{-1} + N^{-1} + qN^{-k})^{\frac{1}{2^{k-1}}}.$$

Theorem 6.2. *Suppose $\alpha \in \mathbb{R}$. Then for every real number $X \geq 1$, there exist $a, q \in \mathbb{Z}$ satisfying $(a, q) = 1$ and $1 \leq q \leq X$ such that*

$$\left|\alpha - \frac{a}{q}\right| \leq \frac{1}{q^X}.$$

Proof. Looking at the $[X]$ numbers

$$\{\alpha t\} = \alpha t - [\alpha t] \text{ with } t = 1, .., [X]$$

and $[X] + 1$ intervals

$$\left[\frac{j-1}{[X]+1}, \frac{j}{[[X]+1]}\right) \text{ for } j = 1, .., [X] + 1,$$

if one of the $\{\alpha t\}$ lies in the interval I_1 or $I_{[X]+1}$ then the theorem holds with $q = t$. If this is not satisfied, then by the pigeonhole principle, one of the $[X] - 1$ intervals must contain two of the $[X]$ numbers. In other words, there exist $t_1, t_2 \in \mathbb{Z}$ with $1 \leq t_1 \leq t_2 \leq [X]$ and an integer $i = 2, \ldots, [X]$ such that $\{t_1\alpha\}, \{t_2\alpha\} \in I_i$ so that

$$|\{t_2\alpha\} - \{t_1\alpha\}| \leq \frac{1}{[X]+1} \leq \frac{1}{X}$$

and subsequently,

$$|(t_2 - t_1)\alpha - ([t_2\alpha] - [t_1\alpha])| \leq \frac{1}{X}.$$

With that, $q = t_2 - t_1$ and $a = [t_2\alpha] - [t_1\alpha])$ will work.

Theorem 6.3. *For every $j = 1, \ldots, k$,*

$$\int_0^1 |f(\alpha)|^{2^j} d\alpha \ll N^{2^j - j + \epsilon}. \tag{6.2}$$

Supposing that $\alpha \in \mathfrak{m}$, using $X = N^k N^{-\nu}$ in Theorem 6.2, there exist $a, q \in \mathbb{Z}$ with $(a, q) = 1$ and $1 \leq q \leq N^k N^{-\nu}$ such that

$$\left|\alpha - \frac{a}{q}\right| \leq \frac{N^\nu}{qN^k}.$$

Considering $\alpha \in \mathfrak{m} \subseteq (N^{\nu-k}, 1-N^{\nu-k})$, $1 \leq a \leq q$. By (6.1),

$$|f(\alpha)| \ll N^{1+\epsilon}(q^{-1}+N^{-1}+qN^{-k})^1/K \tag{6.3}$$
$$\ll N^{1+\epsilon}((N^{\nu})^{-1}+N^{-1}+N^{\nu-1})^1/K$$
$$\ll N^{1+\epsilon-\nu/K},$$

where $K = 2^{k-1}$ for all $\alpha \in \mathfrak{m}$. Combining (6.1)-(6.3),

$$\left|\int_{\mathfrak{m}} f^s(\alpha)e(-\alpha n)d\alpha\right| \ll N^{(1+\epsilon-\nu/K)(s-2^k)N^{2^k-k+\epsilon}} \ll n^{s/k-1-\delta}. \tag{6.4}$$

7. Major Arcs

The goal here is to show that

$$\int_{\mathfrak{M}} f^s(\alpha)e(-\alpha n)d\alpha \gg n^{s/k-1}.$$

Define

$$v(\beta) = \sum_{m=1}^{n} \frac{1}{k} m^{1/k-1} e(\beta m) \tag{7.1}$$

and

$$S(q,a) = \sum_{m=1}^{q} e\left(\frac{am^k}{q}\right). \tag{7.2}$$

Theorem 7.1. *Suppose $a, q \in \mathbb{N}$ satisfy $(a,q) = 1$ and $1 \leq a \leq q \leq N^{\nu}$. Additionally, suppose that $\alpha \in \mathfrak{M}(q,a)$ and*

$$V(\alpha, q, a) = q^{-1}S(q,a)v(\alpha - \frac{a}{q}). \tag{7.3}$$

Then,

$$f(\alpha) = V(\alpha, q, a) + O((N^{\nu})^2).$$

Proof. Write $\beta = \alpha - \frac{a}{q}$. Then,

$$f(\alpha) = \sum_{1\leq x\leq n^{1/k}} e\left(\frac{ax^k}{q}\right) e(\beta x^k)$$
$$= \sum_{m=1}^{n} a(m) e\left(\frac{am}{q}\right) e(\beta m),$$

where

$$a(m) = \begin{cases} 1, & \text{if } m \text{ is a } k-\text{th power,} \\ 0, & \text{if } m \text{ otherwise.} \end{cases}$$

It follows that

$$f(\alpha) - q^{-1}S(q,a)v\left(\alpha - \frac{a}{q}\right)$$
$$= \sum_{m=1}^{n} a_m e(\beta m),$$

where

$$a_m = \begin{cases} e\left(\frac{am}{q}\right) - q^{-1}S(q,a)\frac{1}{k}m^{1/k-1}, & \text{if m is a k- th power,} \\ -q^{-1}S(q,a)\frac{1}{k}m^{1/k-1} & \text{otherwise.} \end{cases}$$

By partial summation,

$$\sum_{m=1}^{n} a_m e(\beta m)$$

$$= e(\beta n)\sum_{m=1}^{n} a_m - 2\pi i\beta \int_0^n e(\beta y)\left(\sum_{m\le y} a_m\right) dy$$

where

$$\sum_{m\le y} a_m = \sum_{x\le y^{1/k}} e\left(\frac{ax^k}{q}\right) - q^{-1}S(q,a)\sum_{m\le y}\frac{1}{k}m^{1/k-1}.$$

Firstly,

$$\sum_{m\le y}\frac{1}{k}m^{1/k-1}$$

$$= \int_1^y \frac{1}{k}x^{1/k-1}dx + O(1)$$

$$= y^{1/k} + O(1).$$

Additionally,

$$\sum_{m\le y} e\left(\frac{ax^k}{q}\right)$$

$$= \sum_{r=1}^{q} e\left(\frac{ar^k}{q}\right)\sum_{\substack{x\le y^{1/k} \\ x\equiv r \bmod q}} 1$$

$$= y^{1/k}q^{-1}S(q,a) + O(q).$$

From this, it follows that

$$\sum_{m\le y} a_m = O(q).$$

Considering that $|\beta| \le N^{\nu-k}$,

$$\sum_{m=1}^{n} a_m e(\beta m) \ll (1+|\beta|n)q \ll (1+PN^{-k}n)P \ll P^2$$

where $P = N^{\nu}$. From this, it follows that,

$$f(\alpha) = V(\alpha,q,a) + O(P^2).$$

8. Partial Summation

Let $z_1, z_2, z_3, \dots$ be a sequence of complex numbers. Suppose further that the function F has continuous derivatives on the interval $[0, X]$. Then,

$$\sum_{m\le X} z_m F(m) = F(X)\sum_{m\le X} z_m - \int_0^X F'(y)\left(\sum_{m\le y} z_m\right)dy. \tag{8.1}$$

Note that

$$F(m) = F(X) - \int_m^X F'(y)dy.$$

Then, if $\alpha \in \mathfrak{M}(q,a)$,

$$f^s(\alpha) - V^s(\alpha,q,a) \ll N^{s-1}|f(\alpha) - V(\alpha,q,a)| \ll N^{s-1}P^2.$$

From this,

$$\sum_{q\le P}\sum_{a=1_{(a,q)=1}}\int_{\mathfrak{M}(q,a)} |f^s(\alpha) - V^s(\alpha,q,a)|d\alpha \ll N^{s-k-1}P^5 \ll n^{s/k-1-1/k}P^5 \ll n^{s/k-1-\delta}$$

for some fixed δ depending on ν. Now it can be written that

$$r(n) = \sum_{q\le P}\sum_{a=1_{(a,q)=1}}\int_{\mathfrak{M}(q,a)} V^s(\alpha,q,a)e(-\alpha n)d\alpha \tag{8.2}$$

then

$$\int_{\mathfrak{M}} f^s(\alpha)e(-\alpha n) = r(n) + O(n^{s/k-1-\delta}). \tag{8.3}$$

With the use of (7.3) and (8.2),

$$\begin{aligned} r(n) &= \sum_{q\le P}\sum_{a=1_{(a,q)=1}}^{q}\int_{\mathfrak{M}(q,a)} (q^{-1}S(q,a))^s v^S\left(\alpha - \frac{a}{q}\right)e(-\alpha n)d\alpha \\ &= \sum_{q\le P}\sum_{a=1_{(a,q)=1}}^{q}\int_{\mathfrak{M}(q,a)} (q^{-1}S(q,a))^s\left(\alpha - \frac{a}{q}\right)e\left(-\left(\alpha - \frac{a}{q}\right)n\right)e\left(-\frac{an}{q}\right)d\alpha \\ &= \sum_{q\le P}\sum_{a=1_{(a,q)=1}} (q^{-1}S(q,a))^s e\left(-\frac{an}{q}\right)\int_{\mathfrak{M}(q,a)} v^s\left(\alpha - \frac{a}{q}\right)e\left(-\left(\alpha - \frac{a}{q}\right)n\right)d\alpha \\ &= \sum_{q\le P}\sum_{a=1_{(a,q)=1}} (q^{-1}S(q,a))^s e\left(-\frac{an}{q}\right)\int_{-PN^{-k}}^{PN^{-k}} v^s(\beta)e(-\beta n)d\beta \\ &= \mathfrak{S}(n,P)\mathfrak{J}^*(n), \end{aligned} \tag{8.4}$$

where

$$\mathfrak{S}(n,P) = \sum_{q\le P}\sum_{a=1_{(a,q)=1}}^{q} (q^{-1}S(q,a))^s e\left(-\frac{an}{q}\right)$$

and

$$\mathfrak{J}^*(n) = \int_{-PN^{-k}}^{PN^{-k}} v^s(\beta)e(-\beta n)d\beta. \tag{8.5}$$

9. Completing the Series

Consider the series $\mathfrak{S}(n, P)$. Write

$$S(q) = \sum_{a=1_{(a,q)=1}}^{q} (q^{-1}S(q,a))^s e\left(-\frac{an}{q}\right). \tag{9.1}$$

Then by (6.1),

$$S(q,a) \ll q^{1+\epsilon-1/K}$$

provided that $(a, q) = 1$. Hence,

$$S(q) \ll q(q^{\epsilon-1/K})^s$$

and so whenever $s \geq 2^k + 1$ and $\epsilon > 0$ is sufficiently small, it can be concluded that

$$S(q) \ll q^{-2^{-k}-1}.$$

Therefore,

$$\mathfrak{S}(n) = \sum_{q=1}^{\infty} S(q) \tag{9.2}$$

converges absolutely and uniformly with respect to n. Additionally,

$$\mathfrak{S}(n, P) - \mathfrak{S}(n) \ll n^{-\delta} \tag{9.3}$$

for some fixed positive number δ. From this,

$$r(n) = (\mathfrak{S}(n) + O(n^{-\delta}))\mathfrak{J}^*(n) \text{ and } \mathfrak{S}(n) \ll 1. \tag{9.4}$$

Theorem 9.1.

$$v(\beta) \ll \min\{n^{1/k}, |\beta|^{-1/k}\}$$

where $|\beta| \leq 1/2$ *and* $\beta \in \mathbb{R}$.

Proof. Suppose that $|\beta| \leq 1/n$. Then,

$$\begin{aligned} v(\beta) &\ll \sum_{m=1}^{M} \frac{1}{k} m^{1/k-1} \\ &= \int_1^n \frac{1}{k} x^{1/k-1} dx + O(1) \ll n^{1/k} \\ &= \min\{n^{1/k}, |\beta|^{-1/k}\}. \end{aligned}$$

Now suppose that $|\beta| > 1/n$. Let $M + [|\beta|^{-1}]$, and

$$v(\beta) = \sum_{m=1}^{M} \frac{1}{k} m^{1/k-1} e(\beta m) + \sum_{m=M+1}^{n} \frac{1}{k} m^{1/k-1} e(\beta m).$$

Next, let

$$S_m = \sum_{r=1}^{m} e(\beta r)$$

and

$$c_m = \frac{1}{k} m^{1/k-1}.$$

Then,

$$\begin{aligned}
&\sum_{m=M+1}^{n} \frac{1}{k} m^{1/k-1} e(\beta m) \\
&= \sum_{m=M+1}^{n} c_m (S_m - S_{m-1}) \\
&= \sum_{m=M+1}^{n} c_m S_m - \sum_{m=M+1}^{n} c_m S_{m-1} \\
&= \sum_{m=M+1}^{n} c_m S_m - \sum_{m=M}^{n=1} c_{m+1} S_m \\
&= c_n S_n - c_{M+1} S_M + \sum_{m=M+1}^{n-1} (c_m - c_{m+1}) S_m.
\end{aligned}$$

Since $S_m \ll |\beta|^{-1}$ and c_m is a decreasing sequence, it follows that

$$\begin{aligned}
&\sum_{m=M+1}^{n} \frac{1}{k} m^{1/k-1} e(\beta m) \\
&\ll c_{M+1} |\beta|^{-1} \\
&< |\beta|^{-1/k} \\
&= \min\{n^{1/k}, |\beta|^{-1/k}\}.
\end{aligned}$$

From this, it follows that

$$v(\beta) \ll \min\{n^{1/k}, |\beta|^{-1/k}\}.$$

Now, let

$$\mathfrak{J}(n) = \int_{-1/2}^{1/2} v^s(\beta) e(-\beta n) d\beta. \tag{9.5}$$

Then,

$$\mathfrak{J}(n) - \mathfrak{J}^*(n) \ll \int_{PN^{-k}}^{1/2} \beta^{-s/k} d\beta \tag{9.6}$$

and

$$\mathfrak{J}(n) \ll \int_{PN^{-k}}^{\infty} \beta^{-s/k} d\beta \ll n^{s/k-1}. \tag{9.7}$$

It follows that

$$\begin{aligned}
r(n) &= (\mathfrak{S}(n) + O(n^{-\delta}))\mathfrak{J}(n) + (\mathfrak{S}(n) + O(n^{-\delta}))(\mathfrak{J}^*(n) - \mathfrak{J}(n)) \\
&= (\mathfrak{S}(n)\mathfrak{J}(n) + O(n^{-\delta}|\mathfrak{J}(n)|) + O(|\mathfrak{J}^*(n) - \mathfrak{J}(n)|) \\
&= \mathfrak{S}(n)\mathfrak{J}(n) + O(n^{s/k-1-\delta})
\end{aligned} \tag{9.8}$$

for some fixed positive number δ. Combining these steps,

$$R(n) = \mathfrak{S}(n)\mathfrak{J}(n) + O(n^{s/k-1-\delta}). \tag{9.9}$$

10. Singular Integral

Theorem 10.1. *Suppose that* $s \geq 2$. *Then,*

$$\mathfrak{J}(n) = \Gamma^s\left(1+\frac{1}{k}\right)\Gamma^{-1}\left(\frac{s}{k}\right)n^{s/k-1} + O(n^{s/k-1/k-1).}$$

The proof of this depends on the following:

Theorem 10.2. *Suppose that* $A, B \in \mathbb{R}$ *satisfy* $A \geq B > 0$ *and* $B \leq 1$. *Then,*

$$\sum_{m=1}^{n-1} m^{B-1}(n-m)^{A-1}$$
$$= \Gamma(B)\Gamma(A)\Gamma^{-1}(B+A)n^{B+A-1} + O(n^{A-1}).$$

Proof. See [Che13].

Proof. (Proof of Theorem 10.2). The function $x^{B-1}(n-x)^{A-1}$ will have no more than one stationary point in the interval $(0, n)$. As a result, the interval $(0, n)$ can be divided into the intervals $(0, X)$ and (X, n), so that $x^{B-1}(n-x)^{A-1}$ is monotonic in each interval. Hence,

$$\sum_{m=1}^{n-1} m^{B-1}(n-m)^{A-1} = \int_0^n x^{B-1}(n-x)^{A-1}dx + O(n^{A-1} + n^{B+A-2})$$
$$= n^{B+A-1}\int_0^1 y^{B-1}(1-y)^{A-1}dy + O(n^{A-1} + n^{B+A-2})$$
$$= \Gamma(B)\Gamma(A)\Gamma^{-1}(B+A)n^{B+A-1} + O(n^{A-1})$$

as required.

Definition 10.3. The gamma function is defined for $t > 0$ by

$$\Gamma(t) = \int_0^\infty e^{-x}x^{t-1}dx.$$

Theorem 10.4. *For every* $A, B > 0$,

$$\Gamma(B)\Gamma(A) = \Gamma(A+B)\int_0^1 y^{B-1}(1-y)^{A-1}dy. \tag{10.1}$$

Proof. Firstly,

$$\Gamma(B)\Gamma(A) = \int_0^\infty e^{-x}x^{B-1}dx \int_0^\infty e^{-x}x^{A-1}dy$$
$$= \lim_{M\to\infty}\int\int_{S_M} e^{-(x+y)}x^{B-1}y^{A-1}dxdy,$$

where S_M denotes the square $[0, M]^2$. Let T_M denote the triangle with vertices $(0, 0)$, $(M, 0)$, and $(0, M)$. Using the substitution $x = u$ and $y = v - u$ and then writing $u = vy$,

$$\int\int_{T_M} e^{-(x+y)}x^{B-1}y^{A-1}dxdy = \int_0^M \left(\int_0^v e^{-v}u^{B-1}(v-u)^{A-1}du\right)dv$$
$$= \int_0^M e^{-v}v^{B+A-1}dv \int_0^1 y^{B-1}(1-y)^{A-1}dy.$$

Considering also that $S_{M/2} \subseteq T_M \subseteq S_M$,

$$\int\int_{S_{M/2}} e^{-(x+y)} x^{B-1} y^{A-1} dxdy \leq \int\int_{T_M} e^{-(x+y)} x^{B-1} y^{A-1} dxdy \leq \int\int_{S_M} e^{-(x+y)} x^{B-1} y^{A-1} dxdy.$$

It follows that

$$\int_0^M e^{-v} v^{B+A-1} dv \int_0^1 y^{B-1}(1-y)^{A-1} dy \to \Gamma(B)\Gamma(A)$$

as $M \to \infty$. Also,

$$\int_0^M e^{-v} v^{B+A-1} dv \to \Gamma(B+A)$$

as $M \to \infty$. The identity (10.1) follows immediately.

11. The Singular Series

Noting that

$$S(q,a) = \sum_{m=1}^{q} e\left(\frac{am^k}{q}\right),$$

the goal is to study the behavior of the series

$$\mathfrak{S}(n) = \sum_{q=1}^{\infty} S(q).$$

The goal is to show that $\mathfrak{S}(n) \gg 1$. This with (9.9) and Theorem 10.1, will show that $R(n) > 0$ for all sufficiently large $n \in \mathbb{N}$.

Definition 11.1. For every prime p, let

$$T(p) = \sum_{h=0}^{\infty} S(p^h).$$

The goal is now to reduce the examination of the series $\mathfrak{S}(n)$ to the study of the series $T(p)$ for finitely many p.

Theorem 11.2. *Suppose that $s \geq 2^k + 1$. Then for every prime p, the series $T(p)$ converges absolutely, the infinite product $\prod_p T(p)$ converges absolutely with*

$$\mathfrak{S}(n) = \prod_p T(p)$$

and there is a positive $C \in \mathbb{R}$, depending on k such that

$$\frac{1}{2} < \prod_{p \geq C} T(p) < \frac{3}{2}.$$

Then the study of the series $\mathfrak{S}(n)$ is reduced to the study of the series $T(p)$ for $p < C$.

Proof. Remembering that $S(q) \ll q^{-1-2^{-k}}$, $T(p)$ converges absolutely. Additionally, if $S(q)$ is multiplicative such that $S(qr) = S(q)S(r)$ whenever $(q,r) = 1$, then

$$\prod_p T(p) = \prod_p \sum_{h=0}^{\infty} S(p^h)$$

$$= \prod_p (1 + S(p) + S(p^2) + \cdots) = \sum_{q=1}^{\infty} S(q)$$

follows as a result of the absolute convergence of $\mathfrak{S}(n)$. The first step is to show that if $(a, q) = (b, r) = (q, r) = 1$, then

$$S(qr, ar + bq) = S(q, a)S(r, b). \tag{11.1}$$

Considering $(q, r) = 1$, as t runs through a complete set of residues modulo q and u runs through a complete set of residues modulo r, $tr + uq$ runs through a complete set of residues modulo qr. Hence,

$$S(qr, ar + bq) = \sum_{m=1}^{qr} e\left(\frac{(ar + bq)m^k}{qr}\right) = \sum_{t=1}^{q} \sum_{u=1}^{r} e\left(\frac{(ar + bq)(tr + uq)^k}{qr}\right)$$

$$= \sum_{t=1}^{q} \sum_{u=1}^{r} e\left(\frac{(ar + bq)(t^k r^k + u^k q^k)}{qr}\right) = \sum_{t=1}^{q} \sum_{u=1}^{r} e\left(\frac{at^k r^k}{q} + \frac{bu^k q^k}{r}\right)$$

$$= \sum_{t=1}^{q} e\left(\frac{at^k r^k}{q}\right) \sum_{u=1}^{r} e\left(\frac{bu^k q^k}{r}\right) = S(q, a)S(r, b).$$

Next, note that as a and b run through sets of residues modulo q and r respectively, $ar + bq$ runs through a reduced set of residues modulo qr. Then due to (11.1),

$$S(qr) = \sum_{\substack{m=1 \\ (m,qr)=1}} ((qr)^{-1} S(qr, m))^s e\left(-\frac{mn}{qr}\right)$$

$$= \sum_{\substack{a=1 \\ (a,q)=1}} \sum_{\substack{b=1 \\ (b,r)=1}} ((qr)^{-1} S(qr, ar + bq))^s e\left(-\frac{(ar + bq)n}{qr}\right)$$

$$= \sum_{\substack{a=1 \\ (a,q)=1}}^{q} (q^{-1} S(q, a))^s e\left(-\frac{an}{q}\right) \sum_{\substack{b=1 \\ (b,r)=1}}^{r} (r^{-1} S(r, b))^s e\left(-\frac{bn}{q}\right)$$

$$= S(q)S(r).$$

The next step is to show the connection between T and the number $M_n(q)$ of solutions to

$$m_1^k + \cdots + m_s^k \equiv n \mod q, \qquad 1 \le m_1, \ldots, m_s \le q.$$

This requires the following result.

Theorem 11.3. *For every prime p,*

$$T(p) = \lim_{\rho \to \infty} p^{\rho(1-s)} M_n(p^\rho).$$

Consider the following, where Theorem 11.3 is the case when $q = p^\rho$ as $\rho \to \infty$.

Theorem 11.4. *For every $q \in \mathbb{N}$,*

$$\sum_{d|p} S(d) = q^{1-s} M_n(q).$$

Proof. Consider that

$$\frac{1}{q}\sum_{u=1}^{q} e\left(\frac{uh}{q}\right) = \begin{cases} 1, & \text{if} q|h, \\ 0, & \text{if} q \nmid h. \end{cases}$$

Using this, it follows that

$$M_n(q) = \sum_{m_1=1}^{q} \cdots \sum_{m_s=1}^{q} \frac{1}{q}\sum_{u=1}^{q} e\left(\frac{u(m_1^k + \cdots + m_s^k - n)}{q}\right)$$

$$= \frac{1}{q}\sum_{u=1}^{q}\sum_{m_1=1}^{q} \cdots \sum_{m_s=1}^{q} e\left(\frac{um_1^k}{q}\right) \cdots e\left(\frac{um_s^k}{q}\right) e\left(-\frac{un}{q}\right)$$

$$= \frac{1}{q}\sum_{u=1}^{q}\left(\sum_{m=1}^{q} e\left(\frac{um^k}{q}\right)\right)^s e\left(-\frac{un}{q}\right).$$

Suppose that

$$\frac{q}{(u,d)} = d. \tag{11.2}$$

Then writing $a = u/(u,q)$ and noting that every $m = 1, \dots, q$ can be of the form $dy + x$ where $y = 0, \dots, q/d - 1$ and $x = 1, \dots, d$ giving

$$\sum_{m=1}^{q} e\left(\frac{um^k}{q}\right) = \sum_{m=1}^{q} e\left(\frac{am^k}{d}\right)$$

$$= \sum_{y=0}^{q/d-1}\sum_{x=1}^{d} e\left(\frac{a(dy+x)^k}{d}\right)$$

$$= \sum_{y=0}^{q/d-1}\sum_{x=1}^{d} e\left(\frac{ax^k}{d}\right)$$

$$= \frac{q}{d}\sum_{x=1}^{d} e\left(\frac{ax^k}{d}\right) = qd^{-1}S(d,a).$$

Now for every $u = 1, \dots, q$, there exists a unique $d|q$ such that (11.2) holds. For this value of d, the condition(11.2) is equivalent tot the condition $(a,d) = 1$. It follows that

$$M_n(q) = \frac{1}{q}\sum_{d|q}\sum_{\substack{a=1\\(a,d)=1}}^{d} (qd^{-1}S(d,a))^s e\left(-\frac{an}{d}\right)$$

$$= q^{s-1}\sum_{d|q}\sum_{\substack{a=1\\(a,d)=1}}^{d} (d^{-1}S(d,a))^s e\left(-\frac{an}{d}\right) = q^{s-1}\sum_{d|q} S(d),$$

as required.

Looking at

$$m_1^k + \cdots + m_s^k \equiv n \mod p^\rho, \qquad 1 \le m_1, \ldots, m_s \le p^\rho \tag{11.3}$$

for all sufficiently large $\rho \in \mathbb{N}$. The goal is to estimate the value of $M_n(p^\rho)$ for larger values of ρ by using estimates of $M_n(p^\rho)$ for smaller values of ρ. In other words, the goal is to find a value γ so that the congruence is solvable for $\rho = \gamma$ with the condition that $(m_1, p^\gamma) = 1$. This additional criteria makes it simpler to obtain lower bounds for $M_n(p^\rho)$ when $\rho \ge \gamma$. It is helpful to recall some things about $k-$th power residues.

Definition 11.5. Suppose that p is a prime and $\rho \in \mathbb{N}$. A number $a \in \mathbb{Z}$ is said to be a $k-$th power residue modulo p^ρ if $p \nmid a$ and the congruence $x^k \equiv a \mod p^\rho$ holds.

The number of $k-$th power residues modulo p^ρ is the number of integer $a \in \mathbb{Z}$ such that $1 \le a \le p^\rho$ and a is a $k-$th power residue modulo p^ρ. Additionally, the number $\nu(p^\rho)$ of $k-$th power residues modulo p^ρ satisfies

$$\nu(p^\rho) = \begin{cases} \frac{\phi(p^\rho)}{(k,\phi(p^\rho))}, & \text{if } p > 2 \text{ or } \rho = 1 \text{ or } k \text{ is odd,} \\ \frac{2^{\rho-2}}{(k,2^{\rho-2})}, & \text{if } p = 2 \text{ and } \rho \ge 2 \text{ and } k \text{ is even.} \end{cases}$$

Suppose $\tau \in \mathbb{Z}$ satisfies

$$p^\tau | k$$

and

$$p^{\tau+1} \nmid k.$$

Define

$$\gamma = \begin{cases} \tau + 1, & \text{if } p > 2 \text{ or } \tau = 0, \\ \tau + 2, & \text{if } p = 2 \text{ and } \tau > 0. \end{cases}$$

Then

$$\nu(p^\gamma) = \frac{\phi(p^{\tau+1})}{(k, \phi(p^{\tau+1}))}.$$

Additionally, the number of solutions to the congruence

$$x^k \equiv a \mod p^\gamma$$

when $p \nmid a$ is either 0 or $p^{\gamma-\tau-1}(k, \phi(p^{\tau_1}))$. Lastly, if a is a $k-$th power residue modulo p^γ, then it is a $k-$th power residue modulo p^ρ for every $\rho \in \mathbb{N}$. For every natural number $q \in \mathbb{N}$, let $M_n^*(q)$ denote the number of solutions to the congruence

$$m_1^k + \cdots m_s^k \equiv n \mod q, \qquad 1 \le m_1, \ldots, m_s \le q, (m_1, q) = 1.$$

Theorem 11.6. *Suppose that $M_n^*(p^\gamma) > 0$. Then for every $\rho \in \mathbb{N}$ with $\rho \ge \gamma$,*

$$M_n(p^\rho) \ge p^{(\rho-\gamma)(s-1)}.$$

Proof. Suppose that

$$x_1^k \equiv n - x_2^k - \cdots - x_s^k \mod p^\gamma,$$

where $1 \le x_1, \ldots, x_s \le p^\gamma$ and $p \nmid x_1$. For every $j = 2, \ldots, s$ there are precisely $p^{\rho-\gamma}$ integers y_j satisfying $1 \le y_j \le p^\rho$ and $y_j \equiv x_j \mod p^\gamma$. It follows that for each pf the $p^{(\rho-\gamma)(s-1)}$ choices of $(s-1)$tuples $(y_2, \ldots, y_s)$, the number $n - y_1^k - \cdots - y_s^k$ is a $k-$th power residue modulo p^γ, and so a $k-$th power residue modulo p^ρ. Hence there exists a y_1 satisfying $1 \le y_1 \le p^\rho$ and $p \nmid y_1$ such that

$${}_1^k \equiv n - y_2^k - \cdots - y_s^k \mod p^\rho.$$

The result follows immediately.

It follows from Theorem 11.4 and (11.6) that if $M_n^*(p^\gamma) > 0$, then

$$T(p) = \lim_{\rho \to \infty} p^{\rho(1-s)} M_n(p^\rho) \geq p^{-\gamma(s-1)}.$$

Then, if this inequality is used for every prime $p < C$, then it follows from Theorem 11.2 that $\mathfrak{S}(n) \gg 1$. It now remains to prove the following.

Theorem 11.7. *Suppose that*

$$s \geq \begin{cases} \frac{p}{p-1}(k, p^\tau(p-1)), & \text{if } p > 2, \\ 2^{\tau+2}, & \text{if } p = 2 \text{ and } k > 2, \\ 5, & \text{if } p = 2 \text{ and } k = 2. \end{cases}$$

Then $M_n^*(p^\gamma) > 0$ *for every* $n \in \mathbb{N}$.

The case when $p = 2$ and $k = 2$ is easy. If $p = 2$ and $k > 2$, then $s \geq 2^\gamma$ and the congruence

$$m_1^k + \cdots + m_s^k \equiv n \mod 2^\gamma, \qquad 1 \leq m_1, \ldots, m_s \leq 2^\gamma, 2 \nmid m_1,$$

can be satisfied by taking m_j to be 0 or 1. When p is odd, make use of the following result.

Theorem 11.8. *Suppose that*

$$X = \{x_1, \ldots, x_r\} \text{ and } Y = \{y_1, \ldots, y_s\}$$

denote r and s incongruent residue classes modulo q respectively. Also suppose that $0 \in Y$ and that $(y_j, q) = 1$ for every $j = 1, \ldots, s$ satisfying $y_j \not\equiv 0 \mod q$. If $X + Y$ denotes the set of residue classes of the form

$$x_i + y + j, \qquad i = 1, \ldots, r, j = 1, \ldots, s,$$

then $\#(X + Y) \geq \min\{q, r + s - 1\}$.

Proof. See [Che13].

12. Proof of Weyl's Inequality and Hua's Lemma

The forward difference operator is as follows. Suppose that ϕ is a real valued function of a real variable. For any $x, h_1 \in \mathbb{R}$, write

$$\Delta_1(\phi(x); h_1) = \phi(x + h_1) - \phi(x).$$

Now denote the jth iterate of the forward difference operator Δ_1 with Δ_j. This means to write

$$\Delta_{j+1}(\phi(x); h_1, \ldots, h_{j+1}) = \Delta_1(\Delta_j(\phi(x); h_1, \ldots, h_j); h_{j+1}).$$

Then for any natural numbers $j \leq k$, the jth iterate Δ_j of the forward difference operator satisfies

$$\begin{aligned} \Delta_j(x^k; h_1, \ldots, h_j) &= \sum_{\substack{\rho_0 \geq 0, \rho_1 \geq 1, \ldots, \rho_j \geq 1 \\ \rho_0 + \rho_1 + \cdots \rho_j = k}} \frac{k!}{\rho_0! \rho_1! \cdots \rho_j!} x^{\rho_0} h_1^{\rho_1} \cdots h_j^{\rho_j} \\ &= h_1 \cdots h_j p_j(x; h_1, \ldots, h_j), \end{aligned}$$

where $p_j(x; h_1, \ldots, h_j)$ is a polynomial in x with integer coefficients, of degree $k - j$ with leading coefficient $\frac{k!}{(k-j)!}$.

Theorem 12.1. *Suppose that*

$$T(\phi) = \sum_{x=1}^{Q} e(\phi(x)),$$

where $\phi : \mathbb{N} \to \mathbb{R}$ is an arithmetic function. Then for any natural number $j \in \mathbb{N}$,

$$|T(\phi)|^{2^j} \leq (2Q)^{2^j - j - 1} \sum_{|h_1|<Q} \cdots \sum_{|h_j|<Q} \sum_{x \in I_j} e(\Delta_j(\phi(x); h_1, \ldots, h_j)),$$

where the intervals $I_j = I_j(h_1, \ldots, h_j)$ satisfy the conditions

$$I_1(h_1) \subseteq [1, Q] \text{ and } I_i(h_1, \ldots, h_j) \subseteq I_{j-1}(h_1, \ldots, h_{j-1}).$$

Proof. The proof uses induction on j. Suppose that $j = 1$. Then,

$$|T(\phi)|^2 = \sum_{x=1}^{Q} \sum_{y=1}^{Q} e(\phi(y) - \phi(x))$$

$$= \sum_{x=1}^{Q} \sum_{h_1=1-x}^{Q-x} e(\phi(x + h_1) - \phi(x))$$

$$= \sum_{h_1=1-Q}^{Q-1} \sum_{x \in I_1}^{Q-x} e(\Delta_1(\phi(x); h_1)) = \sum_{|h_1|<Q} \sum_{x \in I_1} e(\Delta_1(\phi(x); h_1)),$$

where $I_1 = [1, Q] \cap [1 - h_1, Q - h_1]$. Suppose now that the conclusion of the theorem holds for some $j \in \mathbb{N}$ so that

$$|T(\phi)|^{2^j} \leq (2Q)^{2^j - j - 1} \sum_{|h_1|<Q} \cdots \sum_{|h_j|<Q} \sum_{x \in I_j} e(\Delta_j(\phi(x); h_1, \ldots, h_j)).$$

Then by Cauchy's inequality,

$$|T(\phi)|^{2^{j+1}} \leq (2Q)^{2^{j+1} - 2j - 2}(2Q - 1)^j \sum_{|h_1|<Q} \cdots \sum_{|h_j|<Q} \left| \sum_{x \in I_j} e(\Delta_j(\phi(x); h_1, \ldots, h_j)) \right|^2.$$

Then,

$$\left| \sum_{x \in I_j} e(\Delta_j(\phi(x); h_1, \ldots, h_j)) \right|^2 = \sum_{x \in I_j} \sum_{y \in I_j} e(\Delta_j(\phi(y); h_1, \ldots, h_j) - \Delta_j(\phi(x); h_1, \ldots, h_j))$$

$$= \sum_{|h|<Q} \sum_{x \in I_{j+1}} e(\Delta_j(\phi(x + h); h_1, \ldots, h_j) - \Delta_j(\phi(x); h_1, \ldots, h_j))$$

$$= \sum_{|h|<Q} \sum_{x \in I_{j+1}} e(\Delta_{j+1}(\phi(x); h_1, \ldots, h_j); h),$$

where $I_{j+1} = I_j \cap \{x : x + h \in I_j\}$. This gives the conclusion with j replaced by $j + 1$.

Now considering the Theorem 12.1 in the deduction of Theorem 6.1.

Theorem 12.2. *Suppose that $X, Y, \alpha \in \mathbb{R}$ with $X, Y \geq 1$. Suppose further that $|\alpha - a/q| \leq q^{-2}$ with $(a,q) = 1$. Then,*

$$\sum_{x \leq X} \min\{XY^{-1}, ||\alpha x||^{-1}\} \ll XY \left(\frac{1}{q} + \frac{1}{Y} + \frac{q}{XY}\right) \log(2Xq),$$

where $||\beta|| = \min_{n\in\mathbb{Z}} |\beta - n|$ denotes the distance of β to the nearest integer.

Proof. Write

$$S = \sum_{x \leq X} \min\{XY^{-1}, ||\alpha x||^{-1}\}.$$

Every natural number $x \leq X$ can be written in the form $qj + r$, where $j, r \in \mathbb{Z}$ satisfy $0 \leq j \leq X/q$ and $1 \leq r \leq q$. Hence,

$$S \leq \sum_{0 \leq j \leq X/q} \sum_{r=1}^{q} \min\left\{\frac{XY}{qj+r}, ||\alpha(qj+r)||^{-1}\right\}.$$

One can write

$$\begin{aligned}\alpha(qj+r) &= \alpha qj + \frac{ar}{q} + \left(\alpha - \frac{a}{q}\right) r \\ &= \frac{\lfloor \alpha q^2 j \rfloor + \{\alpha q^2 j\}}{q} + \frac{ar}{q} + \frac{(q^2\alpha - qa)r}{q^2} \\ &= \frac{\lfloor \alpha q^2 j \rfloor + ar}{q} + \frac{\{\alpha q^2 j\}}{q} + \frac{(q^2\alpha - qa)r}{q^2}.\end{aligned}$$

Suppose that $j = 0$ and $r \leq q/2$. Then

$$\alpha(qj+r) = \frac{ar}{q} + \frac{(q^2\alpha - qa)r}{q^2}$$

and

$$\left|\frac{(q^2\alpha - qa)r}{q^2}\right| \leq \frac{r}{q^2} \leq \frac{1}{2q}.$$

Thus,

$$||\alpha(qj+r)|| \geq \left|\left|\frac{ar}{q}\right|\right| - \frac{1}{2q} \geq \frac{1}{2}\left|\left|\frac{ar}{q}\right|\right|.$$

Additionally, it is always true that

$$\left|\frac{\{\alpha q^2 j\}}{q} + \frac{(q^2\alpha - qa)r}{q^2}\right| \leq \frac{1}{q} + \frac{r}{q^2} \leq \frac{2}{q}.$$

For every j satisfying $0 \leq j \leq X/q$, as r runs over a complete set of residues modulo q, $\lfloor \alpha q^2 j \rfloor + ar$ also runs through a complete set of residues modulo q. It follows that for any j satisfying $0 \leq j \leq X/q$, there are at most 7 values of r for which the inequality

$$||\alpha(qj+r)|| \geq \frac{1}{2}\left|\left|\frac{\lfloor \alpha q^2 j \rfloor + ar}{q}\right|\right|$$

fails to hold. Additionally note that $qj + r \gg q(j+1)$ if $j \neq 0$ pr $r > q/2$. It follows that

$$S \ll \sum_{1 \le r \le \frac{1}{2}q} \left|\left| \frac{ar}{q} \right|\right|^{-1} + \sum_{0 \le j \le X/q} \left(\frac{XY}{q(j+1)} + \sum_{\substack{r=1 \\ q \nmid \lfloor \alpha q^2 j \rfloor + ar}}^{q} \left|\left| \frac{\lfloor \alpha q^2 j \rfloor + ar}{q} \right|\right|^{-1} \right)$$

$$\ll \frac{XY}{q} \sum_{0 \le j \le X} \frac{1}{j+1} + \left(\frac{X}{q} + 1 \right) \sum_{1 \le h \le \frac{1}{2}q} \left(\frac{h}{q} \right)^{-1}$$

$$\ll \frac{XY}{q} \log(2X) + (X+q)\log q.$$

From this, the theorem follows immediately.

Proof. (Proof of 6.1). Apply Theorem 12.1 with $j = k-1$, $Q = N$, and $\phi(x) = \alpha x^k$ to obtain

$$|f(\alpha)|^K \le (2N)^{K-k} \sum_{|h_1|<N} \cdots \sum_{|h_{k-1}|<N} \sum_{x \in I_{k-1}} e(\Delta_{k-1}(\alpha x^k; h_1, \ldots, h_{k-1})).$$

This gives

$$\Delta_{k-1}(\alpha x^k; h_1, \ldots, h_{k-1}) = k! \alpha h_1 \ldots h_{k-1} \left(x + \frac{h_1}{2} + \cdots + \frac{h_{k-1}}{2} \right),$$

so that

$$|f(\alpha)|^K \le (2N)^{K-k} \sum_{|h_1|<N} \cdots \sum_{|h_{k-1}<N} \sum_{x \in I_{k-1}} e\left(k! \alpha h_1 \cdots h_{k-1} \left(x + \frac{h_1}{2} + \cdots \frac{h_{k-1}}{2} \right) \right).$$

The terms with $h_1 \cdots h_{k-1} = 0$ contribute $\ll N^{k-1}$ to the sum. Hence,

$$|f(\alpha)|^K \ll (2N)^{K-k} \left(N^{k-1} + N^{\epsilon} \sum_{h=1}^{k!N^{k-1}} \min\{N, ||\alpha h||^{-1}\} \right)$$

$$\ll N^{K-k+\epsilon} \left(N^{k-1} + \sum_{h=1}^{k!N^{k-1}} \min\{k!N^k h^{-1}, ||\alpha h||^{-1}\} \right),$$

where the term N^{ϵ} is an upper bound on the number of solutions of the equation

$$k! h_1 \cdots h_{k-1} = h, \quad 0 < |h_1|, \ldots, |h_{k-1}| < N.$$

Now apply Theorem 12.2 with $X = k!N^{k-1}$ and $Y = N$ to obtain

$$\sum_{h+1}^{k!N^{k-1}} \min\{k!N^k h^{-1}, ||\alpha h||^{-1}\} \ll N^k \left(\frac{1}{q} + \frac{1}{N} + \frac{q}{N^k} \right) \log(N^{k-1} q).$$

This gives

$$|f(\alpha)|^L \ll N^{K+2\epsilon}(q^{-1} + N^{-1} + qN^{-k})$$

if $q \le N^k$. Note that the result is trivial if $q > N^k$ so the proof is complete.

Proof. (Proof of 6.3). This proof proceeds by induction on j. Suppose that $j = 1$. Then the integral

$$\int_0^1 |f(\alpha)|^2 d\alpha$$

is equal to the number of solutions of the equation $x^k - y^k = 0$ in natural numbers $x, y \leq N$. There are precisely N solutions. Suppose now the inequality (6.3) holds for some $j \in \mathbb{N}$ satisfying $1 \leq j < k$. Applying Theorem 12.1 with $Q = N$ and $\phi(x) = \alpha x^k$ to obtain

$$|f(\alpha)|^{2^j} \leq (2N)^{2^j - j - 1} \sum_{|h_1|<N} \cdots \sum_{|h_j|<N} \sum_{x \in I_j} e(\Delta_j(\alpha x^k; h_1, \ldots, h_j)),$$

where

$$\Delta_j(\alpha x^k; h_1, \ldots, h_j) = \alpha h_1 \cdots h_j p_j(x; h_1, \ldots, h_j),$$

where $p_j(x; h_1, \ldots, h_j)$ is a polynomial in x of degree $k - j$ with integer coefficients. Hence,

$$|f(\alpha)|^{2^j} \ll (2N)^{2^j - j - 1} \sum_{h \in \mathbb{Z}} c_h e(\alpha h), \tag{12.1}$$

where for every $h \in \mathbb{Z}$, c_h denotes the number of solutions of the equation

$$h_1 \cdots h_j p_j(x; h_1, \ldots, h_j) = h, \qquad |h_1|, \ldots, |h_j| < N, x \in I_j,$$

Note that $c_0 \ll N^j$ and $c_h \ll N^\epsilon$ if $h \neq 0$. Additionally, one can write

$$\begin{aligned} |f(\alpha)|^{2^j} &= (f(\alpha))^{2^{j-1}} (f(-\alpha))^{2^{j-1}} \\ &= \sum_{h \in \mathbb{Z}} b_h e(-\alpha h), \end{aligned} \tag{12.2}$$

where, for every $h \in \mathbb{Z}$, b_h denotes the number of solutions of the equation

$$x_1^k + \cdots x_{2^{j-1}}^k - y_1^k + \cdots - y_{2^{j-1}}^k = h, \qquad 1 \leq x_1, \ldots, x_{2^{j-1}}, y_1, \ldots, y_{2^{j-1}} \leq N.$$

Hence,

$$\sum_{h \in \mathbb{Z}} b_h = |f(0)|^{2^j} = N^{2^j}.$$

Then by the induction hypothesis,

$$b_0 = \int_0^1 |f(\alpha)|^{2^j} d\alpha \ll N^{2^j - j + \epsilon}.$$

It follows from (12.1) and (12.2) and Parseval's identity that

$$\int_0^1 |f(\alpha)|^{2^{j+1}} d\alpha \ll (2N)^{2^j - j - 1} \sum_{h \in \mathbb{Z} c_h b_h} .$$

Now note that

$$\sum_{h \in \mathbb{Z}} c_h b_h \ll c_0 b_0 + N^\epsilon \sum_{h \neq 0} b_h \ll N^j N^{2^j - j + \epsilon} + N^\epsilon N^{2^j},$$

so that

$$\int_0^1 |f(\alpha)|^{2^{j+1}} d\alpha \ll N^{2^{j+1} - j - 1 + \epsilon}.$$

This completes the proof.

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