J-Functions

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- Dedekind, Einsteinian, Fourier
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What are J-Functions?

J functions parameterize elliptic curves and are also modular functions of weight zero of the upper half of the complex plane.

Klein J-Invariant

A J-Function is holomorphic (part of complex analysis), a complex valued function that is differentiable every point of the domain in the complex plane C^n

Example

$j(\tau) = 1728\frac{g_2(\tau)^3}{\Delta(\tau)} = 1728\frac{g_2(\tau)^3}{g_2(\tau)^3 - 27g_3(\tau)^2} = 1728\frac{g_2(\tau)^3}{(2\pi)^{12}\eta^{24}(\tau)}$ (τ is an arbitrary complex variable of the upper half of the complex plane.)

Expressions of J-Tao

Remark

Now, you may be wondering what's all these variables listed above... J-functions aren't only just by themselves; they are composed of other functions such as Dedekind, Einsteinian, and Fourier.

$\Delta(\tau) = g_2(\tau)^3 - 27g_3(\tau)^2 = (2\pi)^{12}\eta^{24}(\tau)$ where $\eta(\tau) = e^{\frac{\pi i\tau}{12}} \prod_{n=1}^{\infty}\left(1 - e^{2n\pi i\tau}\right) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1 - q^n)$ such that $q = e^{2\pi i\tau}$ (Dedekind eta function) Now, let's decompose $g_2(\tau)$...

$$g_2(\tau) = 60\,G_4(\tau)$$

(What's $G_4(\tau)$?)

$$G_4(\tau) = \frac{\pi^4}{45} E_4(\tau)$$

$$E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n}$$

Or we could also express j-functions in terms of Einsteinian functions as such that $j(\tau) = 1728 \frac{E_4(\tau)^3}{E_4(\tau)^3 - E_6(\tau)^2}$. We have previously defined $E_4(\tau)$. Here, $E_6(\tau) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1-q^n}$

Proof that $j(\frac{1+i\sqrt{163}}{2})$

is an integer, approx. -262537412640768000

Q Series:

$$j(\tau) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \dots, \quad q = e^{2\pi i\tau}$$

$$j(\frac{1+i\sqrt{163}}{2}) = -e^{\pi\sqrt{163}} + 744 + 196884e^{\pi\sqrt{163}} + \dots$$

Interesting Fact

196884= 1 trivial representation + 196883 irreducible representations (the dimensions that Robert Griess used to construct the Monster)

Simple Reasoning

Back to the original formula for $j(\tau) = 1728\frac{g_2(\tau)^3}{\Delta(\tau)}$

$$g_2(\tau)$$

has weight 4, and

$$g_2(\tau)^3$$

has weight 12
and the discriminant

$$\Delta(\tau)$$

has weight 12
Thus, $j(\tau)$ has weight 0 with both "weights canceling"

Special J-Values

Examples

$J(i) = J\left(\frac{1+i}{2}\right) = 1$

$J(\sqrt{2}i) = \left(\frac{5}{3}\right)^3$

$J(2\ i) = \left(\frac{11}{2}\right)^3$

$$J\left(\frac{\sqrt{30}i}{1}\right) = \frac{1}{4}\left(7+5\sqrt{2}+3\sqrt{5}+2\sqrt{10}\right)^4\left(55+30\sqrt{2}+12\sqrt{5}+10\sqrt{10}\right)^3$$

$$J\left(\frac{\sqrt{30}i}{2}\right) = \frac{1}{4}\left(7+5\sqrt{2}-3\sqrt{5}-2\sqrt{10}\right)^4\left(55+30\sqrt{2}-12\sqrt{5}-10\sqrt{10}\right)^3$$

$$J\left(\frac{\sqrt{30}i}{5}\right) = \frac{1}{4}\left(7-5\sqrt{2}+3\sqrt{5}-2\sqrt{10}\right)^4\left(55-30\sqrt{2}+12\sqrt{5}-10\sqrt{10}\right)^3$$

$$J\left(\frac{\sqrt{30}i}{10}\right) = \frac{1}{4}\left(7-5\sqrt{2}-3\sqrt{5}+2\sqrt{10}\right)^4\left(55-30\sqrt{2}-12\sqrt{5}+10\sqrt{10}\right)^3$$

$$J\left(\frac{1+\sqrt{31}i}{2}\right) = \left(1-\frac{1}{64}\left[(13+\sqrt{93})\sqrt[3]{4(29-3\sqrt{93})}+(13-\sqrt{93})\sqrt[3]{4(29+3\sqrt{93})}+8\right]^2\right)^3$$

$$J(\sqrt{70}i) = \left(1+\frac{9}{4}\left(303+220\sqrt{2}+139\sqrt{5}+96\sqrt{10}\right)^2\right)^3$$

$$J(7i) = \left(1+\frac{9}{32}\sqrt[4]{28}\left(3+\sqrt{7}\right)^3\left(13+3\sqrt{7}+\left(6+\sqrt{7}\right)\sqrt[4]{28}\right)^2\right)^3$$

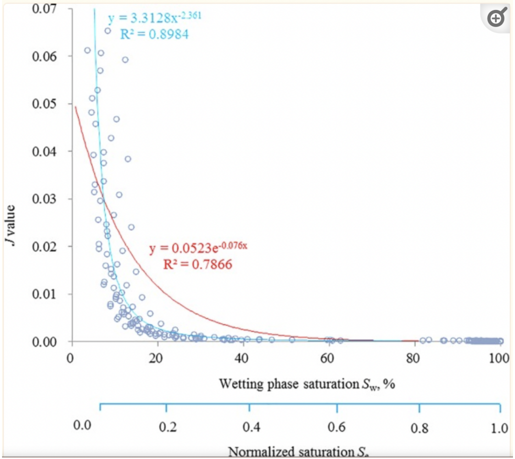
$$J(8i) = \left(1+\frac{9}{4}\sqrt[4]{2}\left(1+\sqrt{2}\right)\left(123+104\sqrt[4]{2}+88\sqrt{2}+73\sqrt[4]{8}\right)^2\right)^3$$

$$J(10i) = \left(1+\frac{9}{8}\left(2402+1607\sqrt[4]{5}+1074\sqrt[4]{25}+719\sqrt[4]{125}\right)^2\right)^3$$

$$J\left(\frac{5i}{2}\right) = \left(1+\frac{9}{8}\left(2402-1607\sqrt[4]{5}+1074\sqrt[4]{25}-719\sqrt[4]{125}\right)^2\right)^3$$

$$J(2\sqrt{58}i) = \left(1+\frac{9}{256}\left(1+\sqrt{2}\right)^5\left(5+\sqrt{29}\right)^5\left(793+907\sqrt{2}+237\sqrt{29}+103\sqrt{58}\right)^2\right)^3$$

Applications: Leverett J-Function

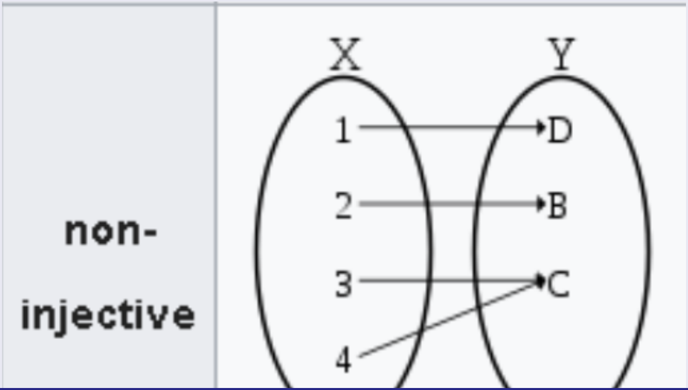


Remark: The J-Prediction Formula: $J(S_e) = \frac{p_c(S_e)}{\sigma \cos\theta}\sqrt{\frac{k_{av}}{\phi_{av}}} = A(S_e)^B$

Surjective

Theorem of Surjectivity

A surjective function is a function f such that every element y can be mapped from some element x so that $f(x) = y$.



Elliptic Curves and J-functions

Weierstrass Form:

$$y^2 = x^3 + ax + b$$

J-invariant

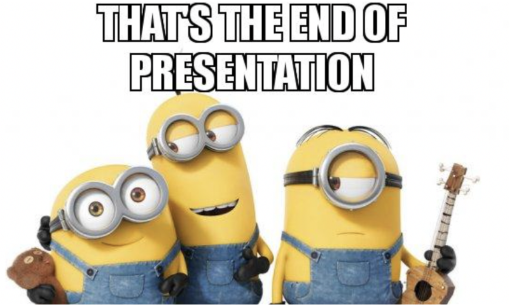
$$j(E) = 1728 \frac{4a^3}{4a^3 + 27b^2}$$

E for Elliptic Curve

Theorem

Let E, E0 be elliptic curves over Q. Then E $\cong E'$ over C iff $j(E) = j(E')$. Given the field K and elliptic curves E, E0 over K then E $\cong E'$ over K iff $j(E) = j(E')$.

Questions?



References

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