Jay Bird: An Expository Article on J-Functions

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0.1 Acknowledgements

I acknowledge the ones who are idiotic but mathematically empowered.

0.2 Abstract

J-Functions parameterize elliptic curves, and they are modular functions of weight 0 of the upper half of the complex plane C. What's more, J-invariants connect number theory with the Monster Group, which is shown in q-expansion with 196884 as the coefficient of the third term. J-functions was the foundation for Andrew Wiles to prove one of the Millennium problems: Fermat's Last Theorem. This paper will cover the basics of J-Functions: Function properties, Q-expansion, proofs of why $e^{\pi\sqrt{163}}$ is close to an integer, and its connection with elliptic curves.

0.3 Introduction

J-Functions, in other words the Klein's function not to be mixed up with other functions such as the Leverett J-Function, are composed of functions such as Fourier, eta-Dedekind functions. The J-Function:

$$J(\omega_1, \omega_2) = \frac{g_2^3(\omega_1, \omega_2)}{\Delta(\omega_1, \omega_2)} = \frac{\lambda^{-12} g_2^3(\omega_1, \omega_2)}{\lambda^{-12}\Delta(\omega_1, \omega_2)} = \frac{g_2^3(\lambda\omega_1, \lambda\omega_2)}{\Delta(\lambda\omega_1, \lambda\omega_2)} = J(\lambda\omega_1, \lambda\omega_2)$$

Another Expression:

$$j(\tau) = 1728 \cdot \frac{g_2(\tau)^3}{g_2(\tau)^3 - 27g_3(\tau)^2}$$

$$g_2(\tau) = 60G_4(\tau)$$
$$g_3(\tau) = 140G_6(\tau)$$
$$G_4(\tau) = \frac{\pi^4}{45}E_4(\tau)$$
$$G_6(\tau) = \frac{2\pi^6}{945}E_6(\tau)$$
$$E_4(\tau) = 1 + 240\sum_{n=1}^{\infty}\frac{n^3 q^n}{1-q^n}$$
$$E_6(\tau) = 1 - 504\sum_{n=1}^{\infty}\frac{n^5 q^n}{1-q^n}$$

where $q = e^{2\pi i\tau}$ Note: $\tau \in \mathbf{C}$

0.3.1 Why the J-Functions has weight 0

$$j(\tau) = 1728 \cdot \frac{g_2(\tau)^3}{g_2(\tau)^3 - 27g_3(\tau)^2}$$

$$j(\tau) = 1728\frac{g_2(\tau)^3}{\Delta(\tau)}$$

$g_2(\tau)$ itself has weight 4, and $g_2(\tau)^3$ has weight 12, which makes up the numerator. The discriminant for the denominator has weight 12. Holistically, the j-function itself has weight 0.

$g_2(\tau)$ and $g_3(\tau)$ are infinite sums over points of a lattice.

0.4 Q-expansion

$$j(\tau) = \frac{1}{q}+744+196884q+21493760q^2+864299970q^3+\cdots := \frac{1}{q}+\sum_{n=0}^{\infty} c(n)q^n$$

where τ is a complex variable of the upper-half of the complex plane. If we plug in $q = e^{\pi\sqrt{163}}$ to the q-series, we get a negative number that is very close to the an integer, approx. -262537516407680000, satisfying a complex variable τ that equals to $\frac{1+\sqrt{-163}}{2}$. Plugging in $q = e^{\pi\sqrt{163}}$ gives us an integer as such that we could disregard the terms after the 2nd or 3rd term since the output numbers are infinitely small.

0.4.1 The Monster

196884 was the dimension that Robert Griess used to construct the Monster. Conway and Norton noted in the Monstrous Moonshine that...

$$196884 = 196883 + 1$$

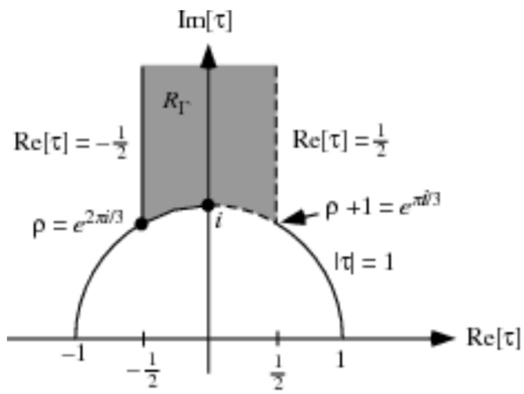
$$21493760 = 21296876 + 196883$$

$$864299970 = 842609326 + 2 \cdot 21296876 + 2 \cdot 196883 + 2 \cdot 1$$

0.4.2 Special J-Values

$$J(i) = J\left(\frac{1+i}{2}\right) = 1 J(\sqrt{2}i) = \left(\frac{5}{3}\right)^3 J(2i) = \left(\frac{11}{2}\right)^3 J(2\sqrt{2}i) = \frac{125}{216}(19+13\sqrt{2})^3$$

0.5 Regions



The J-function is a holomorphic (to be defined afterwards) function $\mathbf{H} \to \mathbf{C}$, taking on every value of the upper half of the complex plane exactly once when restricted to the shaded region from the figure above.

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}, \quad ad - bc = 1$$

$$|\tau| \geq 1$$

$$-\frac{1}{2} < \Re(\tau) \leq \frac{1}{2}$$

0.6 Symbols

$\mathrm{SL}(2, \mathbf{R})$, which is a two by two matrix with determinant one, acts on the upper half of the complex plane C as shown in the operation below.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d}$$

0.6.1 Holomorphism

Holomorphic functions are extensively studied in complex analysis. J-Functions are holomorphic, meaning that they are complex differentiable on C^n

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

This is similar to the real function when we have limits, except that the input value is a complex variable in this case.

0.6.2 Definition

In complex analysis, if a function is holomorphic, then the function is continuous. Moreover, if we let j be a function, then it is holomorphic iff it is complex differentiable everywhere on U. Interestingly, 'Holo' means whole and 'morphic' means shape.

0.6.3 Complex Functions

A complex function is holomorphic iff they have a convergent power series expansion at each point on C. For example, we could observe from the q-expansion for j-functions that the terms n or convergent, or getting smaller as n approaches to infinity.

Basic Operations

$$f + g$$
$$f - g$$
$$fg$$
$$f/g$$

These four operations listed above makes the function holomorphic. And also the composite function as well in the complex plane.

$$a_0 + a_1 z + a_2 z^2 \cdot$$

The following polynomial with the complex variable converges iff $|z| < R$

0.6.4 Revisit

Definition: A function is holomorphic on an open set U if it is complex differentiable on every point of U.

Remark:

According to [0.6] and [0.6.1], j-invariants are defined to be holomorphic after $\mathrm{SL}(2, \mathbf{R})$ acts on the complex plane C of a linear transformation.

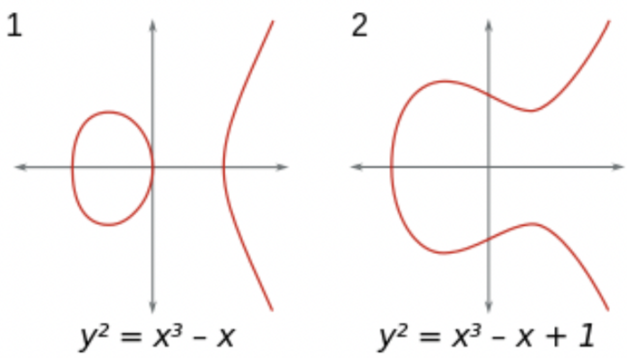
0.7 J-Functions and Elliptic Curves

$$y^2 = x^3 + ax + b$$

$$\Delta = -16\left(4a^3 + 27b^2\right) \neq 0$$

0.8 Definition

An elliptic curve is also called the Weierstrass equation, and it does not have cusps, intersections, nor isolated points iff the discriminant that does not equal to 0.



$$j(E) = 1728\frac{4a^3}{4a^3 + 27b^2}$$

Definition

Theorem

Let E, E0 be elliptic curves over Q. Then E $\cong E'$ over C iff $j(E) = j(E')$. Given the field K and elliptic curves E, E0 over K then E $\cong E'$ over K iff $j(E) = j(E')$.

The above definition explains the fact of why j-functions are function to parameterize elliptic curves. The following theorem states that two elliptic curves are **isomorphic** if and only if their j-invariant is equivalent.

0.9 J-invariants and Elliptic Curves

Another form of the elliptic function is the Legendre form.

$$y^2 = x(x-1)(x-\lambda)$$

The j-inv. in this case is...

$$j = 256\frac{\left(\lambda^2 - \lambda + 1\right)^3}{\lambda^2(\lambda-1)^2}$$

Remark:

Definition of **Homomorphism**: A mapping of f from a group (G, o) to another group (G', o') satisfying...

$$f(aob) = f(a)o'f(b)$$

satisfying

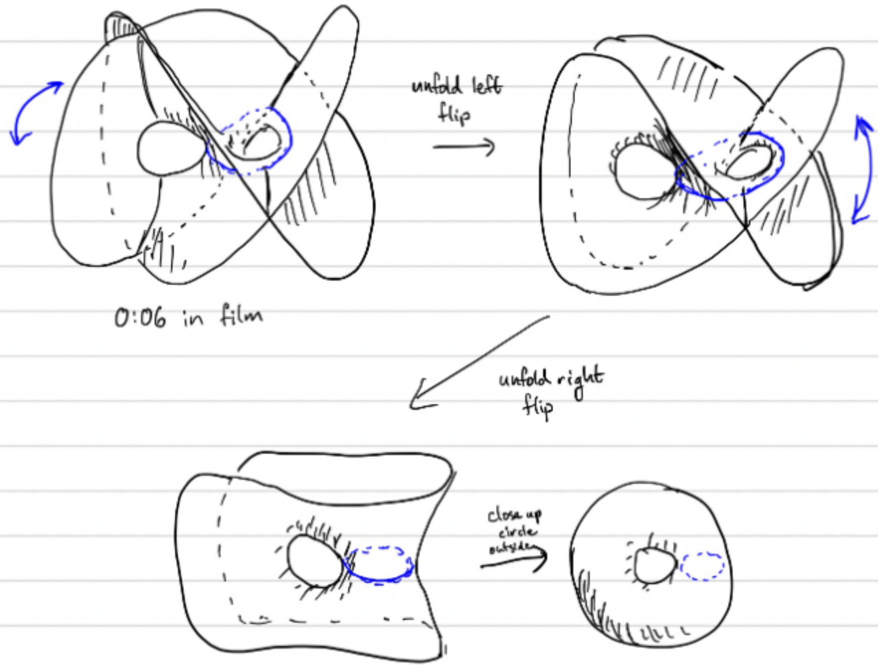
$$a, b \in G$$

$$f(a), f(b) \in G'$$

Definition of **Isomorphism**: 1. f is Homomorphic 2. f is onto 3. f is one-to-one

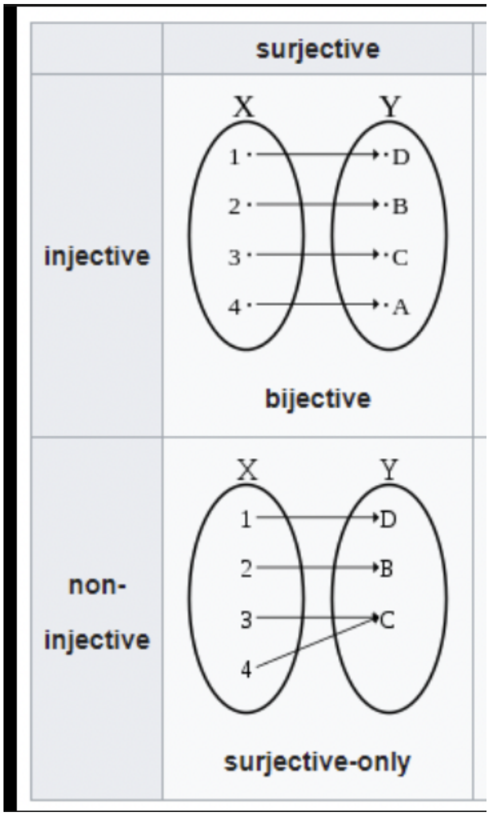
An Elliptic curve is isomorphic to a torus topologically speaking. In other words, elliptic curves are equivalent to a torus in the complex plane.

Seeing the torus:



0.10 J-Functions are surjective

0.10.1 Proof



$$j : H \to C$$

The diagram from above illustrates this as such that at least one element in the codomain has to be mapped to an element in the domain.

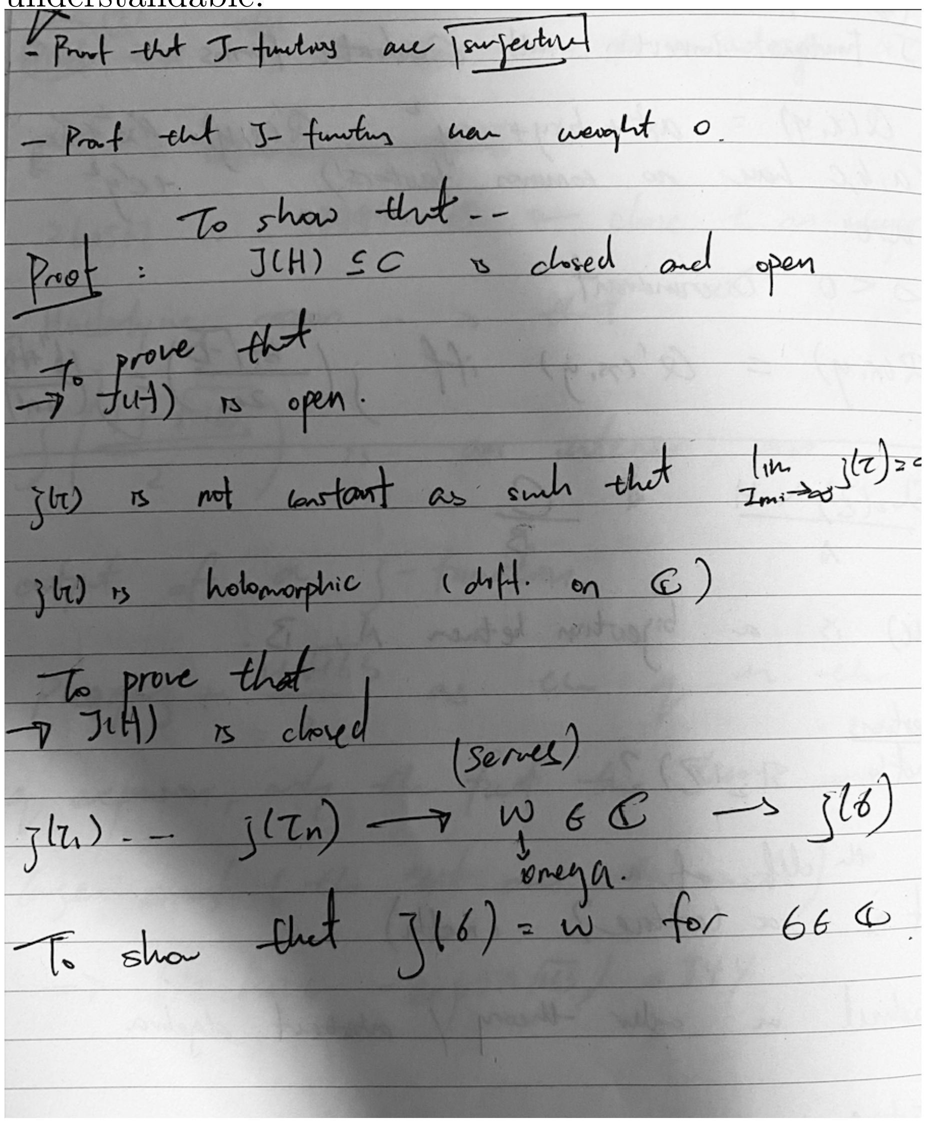
Theorem: The j-invariant is surjective

Proof:

0.10.2 Continued

[Proof]

Here's a graphic form of my proof, hopefully visually understandable.



- Proof that J-functions are surjective

- Proof that J-function have weight 0.

To show that --

Proof : $J(H) \subseteq C$ is closed and open

To prove that

→ $J(H)$ is open.

$j(\tau)$ is not constant as such that $\lim_{Im \tau \to \infty} j(\tau) = \infty$

$j(\tau)$ is holomorphic (diff. on $\mathbb{C}$)

To prove that

→ $J(H)$ is closed

(series)

$j(\tau_1)$ -- $j(\tau_n) \longrightarrow \omega \in \mathbb{C} \longrightarrow j(\sigma)$

ω ↓ omega.

To show that $j(\sigma) = \omega$ for $\sigma \in \mathbb{C}$.

0.10.3 Continued

Explanation: In order to prove that j-functions are surjective, we have to show that the j-function is 'closed' and 'open'. First and foremost, in order to prove that the j-function is open, we have to prove that j(H) is non-constant (H is a complex variable of the upper-half of the complex plane). As illustrated in my graphic, to prove that j(H) is non-constant, we have to show that the limit of the imaginary terms approaching infinity of j(H) is infinity. However, not only does proving the limit suffices, but also proving that j(H) is holomorphic, meaning that it is differentiable on every point of the complex plane. We prove this by understanding the definition of holomorphic functions (complex differentiable as $z \to z_0$). For the latter part, we still need to prove that $j(H)$ is closed. According to my graphic illustration above, we have the series $j(\tau_1), j(\tau_2), \ldots \to \omega$ (omega is part of the complex plane) that spans infinitely to the nth j(tao) term. We have omega (w) and sigma to be part of the complex plane. To prove that j(H) is closed, we need to show that J(sigma)=omega. Another note here: for the j(tao) series that we have to prove that it converges to omega, we need to incorporate the Bolzano-Weierstrass theorem.

0.11 J-Functions and the Monster

Previously, I have mentioned that J-functions offer a link between the number theory field and also Group Theory, specifically the Monster. The connection between the J-Function and the Monster is known as a special case of the Monstrous Moonshine. Interestingly, the 'Moonshine' term was specifically coined by Professor Andrew Ogg, a Emeritus professor Berkeley, with a celebration of a bottle of whiskey—facts stated according to Professor Robin Hartshorne's personal note, whose also a Emeritus Professor at Berkeley (Robin Hartshorne, my podcast candidate).

$$V_\natural = \bigoplus_{n=-1}^{\infty} V_n$$

The equation listed above is the Moonshine Module that Igor Frenkel, James Lepowsky, and Arne Meurman constructed in 1988. The equation represnts the sum of representations of the Monster. Interestingly, the dimensions of each V_n aligns with the coefficients of the q-expansion in j-invariants. Conway and Norton conjectured that "the existence of an infinite dimensional graded representation of M, whose graded traces Tg are the expansions of precisely the functions on their list." Despite the fact that the Conway, Norton conjecture existed for a while, Richard Borcherds has eventually proved the conjecture (listed on the next page).

0.12 Continued

Theorem [Borcherds]: Let $V = V_\natural = \oplus_{n \in Z}^{\infty} V_n$ be a Moonshine Module constructed by Frenkel, Lepowsky, and Meurman. For any $g \in M$, the McKay-Thompson series

$$T_g(q) = \sum_{n=-1}^{\infty} tr\,(g \mid V_n)\, q^n$$

is a hauptmodul for a genus 0 subgroup of SL2(R).

If we let g=1, the identity of the Monster, then each of the coefficients of the j-function after Fourier Expansion aligns with the representations of V_n.

$$T_1(q) = \frac{1}{q} + 196884q + 2149360q^2 + \cdots = j(\tau) - 744$$

Now, you may be wondering. How did Borcherds prove this?

Borcherds incorporated the fundamental properties of the j-function in order to prove the Moonshine.

Borcherds' Theorem: $p = e^{2\pi iz}$ and $q = e^{2\pi i\tau}$ for $z, \tau \in H$

$$p^{-1} \prod_{m>0, n \in Z} (1 - p^m q^n)^{c(mn)} = j(p) - j(q)$$

$$j(q) - 744 = \sum_{n=-1}^{\infty} c(n) q^n = q^{-1} + 196884q + \cdots$$

$j(p) := j(z)$ $j(q) := j(\tau)$

Proof:

Multiplying the left side of Borcherds' equation by p and taking the natural logarithmn...

$$\ln\left(\prod_{m>0,n\in Z}(1-p^m q^n)^{c(mn)}\right)$$

$$=\sum_{m>0}\sum_{n\in Z}\ln(1-p^m q^n)^{c(mn)}$$

$$=\sum_{m>0}\sum_{n\in Z}c(mn)\ln(1-p^m q^n)$$

After Laurent Expansion...

$$\sum_{m>0}\sum_{n\in Z}\sum_{k>0}-c(mn)\frac{(p^m q^n)^k}{k}=-\sum_{m>0}\sum_{n\in Z}\sum_{k>0}c(mn)\frac{p^{mk}q^{nk}}{k}$$

Let $m_0=mk$ and $n_0=nk$

$$\sum_{m>0}\sum_{n\in Z}\sum_{k>0}c(mn)\frac{p^{m_0}q^{n_0}}{k}$$

There are series of simplifying and other operations, which I will not state in this paper due to its longevity and complication. However, we arrive to be at this equation after applying m-th Hecke operator, the linear operator, and Fourier Expansion to the left hand side of Borcherds' formula.

$$\sum_{m\geq -1}g_m(j(q))p^m$$

(Left hand side of Borcherds' formula)

0.13 Continued

The ultimate step that we want to come forth to is to show that

$$j(p) - j(q) = \frac{1}{p} + 744 + \sum c(m)p^m - j(q)$$

(right hand side of Borcherds' formula)

as such that the right hand side is equivalent to the left hand side of Borcherds' formula.

Using the binomial theorem and further simplifications to manipulate the right hand side, we ultimately show that the right and left hand side of Borhcerds' theorem is equivalent as the polynomials of $j(q)$ are equal.

Note: The actual proof is notably profound. I have only illustrated a brief picture of what Borcherds' proof looks like.

Works Cited

Wikipedia contributors. "J-invariant." Wikipedia, The Free Encyclopedia. Wikipedia, The Free Encyclopedia, 5 Jun. 2023. Web. 16 Jul. 2023.

Piezas, Tito III and Weisstein, Eric W. "j-Function." From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/ Function.html

Scherer, Asa. "The j-Function and the Monster." (2010).

Riley, Catherine. "Monstrous Moonshine: The Monster and the Elliptic Modular J-function." (2022).

Klarreich, Erica, and Mircea Pitici. "Mathematicians chase moonshine's shadow." The Best Writing on Mathematics (2016).

Wikipedia contributors. "Monstrous moonshine." Wikipedia, The Free Encyclopedia. Wikipedia, The Free Encyclopedia, 10 Jun. 2023. Web. 16 Jul. 2023.