On Phase Transitions of Special Graph Properties in Random Graphs

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Euler Circle

July 15, 2023

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w.h.p.: $\lim_{n\to\infty} \mathbb{P}(\mathscr{A}_{\ltimes}) = 1$

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Chebyshev + Cauchy: $\mathbb{P}(X = 0) \leq \frac{VarX}{\mathbb{E}X^2} = 1 - \frac{(\mathbb{E}X)^2}{\mathbb{E}X^2}$

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For every two disjoint finite sets of vertices, U and V, there exist a vertex x outside of U and V that is connected to all vertices in U but contains no neighbors of V.

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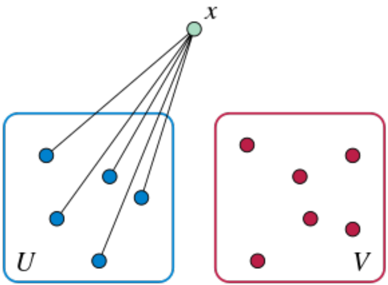

Proof of Extension Proeprty

Proof.

Let there be x_1 vertices in U and x_2 vertices in V. Then the probability some vertex x exists is $p^{x_1}(1-p)^{x_2}$. Due to there being an infinite number of vertices, the probability that no x exists is $[1-p^{x_1}(1-p)^{x_2}]^{\infty}$ which is 0, so some x exists. ■

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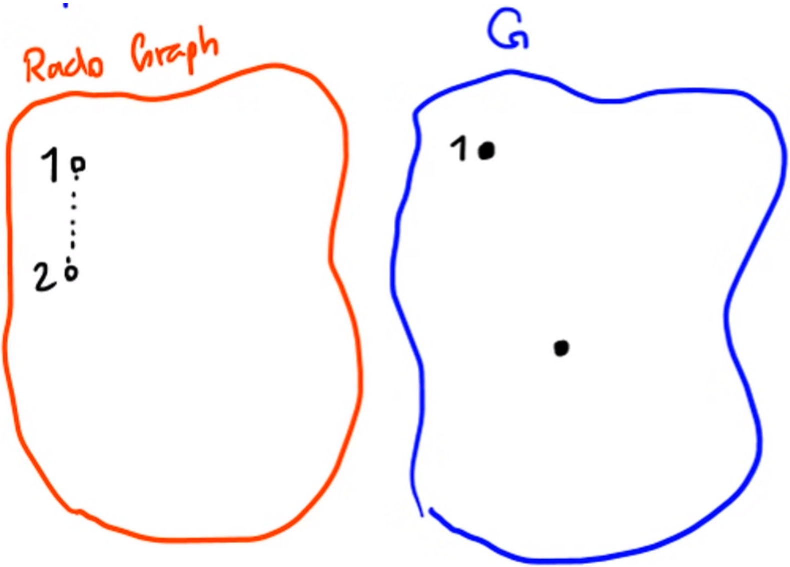
4. Repeat

Visualization



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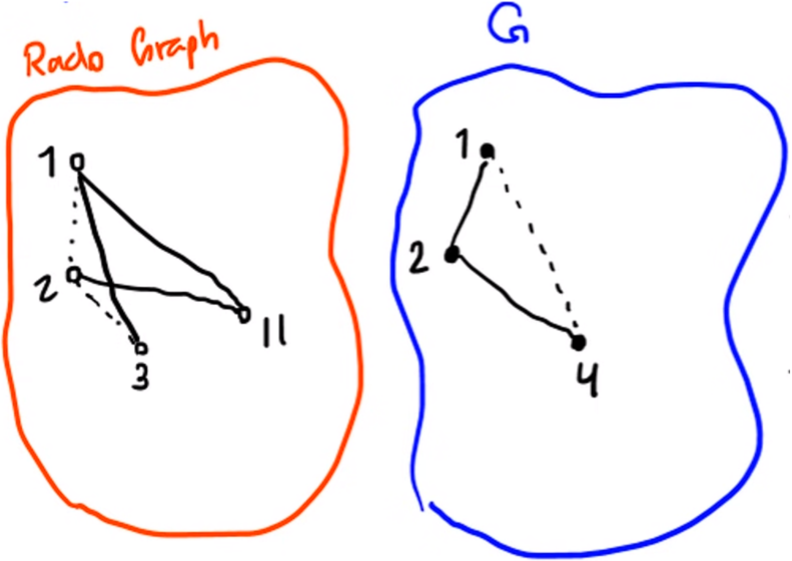

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Lemma

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Proof.

We know that there are a total of $\binom{\binom{n}{2}}{m}$ graphs containing n vertices and m edges. We also know that

$\mathbb{P}(\mathbb{G}_{n,p} = G_0 | e(\mathbb{G}) = m) = \frac{p^m(1-p)^{\binom{n}{2}-m}}{\binom{\binom{n}{2}}{m} p^m (1-p)^{\binom{n}{2}-m}} = \binom{\binom{n}{2}}{m}^{-1}$ This means that each graph will show up with equal probabilities. ∎

Monotone Graphs

Definition

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(Monotone Decreasing) A graph property to be monotone decreasing if $G \in \mathscr{P}$ and if we remove any edge, our new graph still contains this property.

Relating $\mathbb{G}_{n,m}$ with $\mathbb{G}_{n,p}$ Continued

Lemma

Let $\mathscr{P}$ be any graph property. Then when $p = \frac{m}{\binom{n}{2}}$ where $m = m(n) \to \infty$, and $\binom{n}{2} - m \to \infty$. Then for large n,

$$\mathbb{P}(\mathbb{G}_{n,m} \in \mathscr{P}) \leq 10m^{1/2}\mathbb{P}(\mathbb{G}_{n,p} \in \mathscr{P}) \tag{0.1}$$

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Lemma

Let $\mathscr{P}$ be any monotone increasing graph property with $p = \frac{m}{N}$ Then for large n and $p = o(1)$ such that $Np, N(1-p)/(Np)^{1/2} \to \infty$,

$$\mathbb{P}(\mathbb{G}_{n,m} \in \mathscr{P}) \leq 3\mathbb{P}(\mathbb{G}_{n,p} \in \mathscr{P}) \tag{0.2}$$

Thresholds

Definition

A function $m^* = m^*(n)$ is a threshold for a monotone increasing graph property $\mathscr{P}$ in a random graph $\mathbb{G}_{n,m}$ if

$$\lim_{x\to\infty} \mathbb{P}(\mathbb{G}_{n,m} \in \mathscr{P}) = \begin{cases} 0 & \text{if } m/m^* \to 0 \\ 1 & \text{if } m/m^* \to \infty \end{cases} \quad (0.3)$$

as $n \to \infty$

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Applying our Theorems and Lemmas

Theorem

(Bollobàs and Thomason) Every non-trivial monotone graph property has a threshold. (Not Proved Here)

Theorem

Let $\mathscr{P} = \{$*non edge-less sets of labeled graphs* $\mathbb{G}_{n,p}\}$ *Then*

$$\lim_{x\to\infty} \mathbb{P}(\mathbb{G}_{n,p} \in \mathscr{P}) = \begin{cases} 0 & \text{if } p \ll n^{-2} \\ 1 & \text{if } p \gg n^{-2} \end{cases} \tag{0.5}$$

Theorem

If $m/n \to \infty$ *then w.h.p.* $\mathbb{G}_{n,m}$ *contains at least one triangle. (Not Proved Here)*

Proof of Threshold for having edge

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$$\mathbb{P}(X > 0) \leq \frac{n^2}{2}p \to 0 \quad (0.6)$$

when $p \ll n^{-2}$ for $n \to \infty$ by First Moment Method.

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Now we show: $\operatorname{Var} X \approx \mathbb{E}X$ w.h.p. because the probability will approach 1 when $\operatorname{Var} X/(\mathbb{E}X)^2 \to 0$.

$$\operatorname{Var} X/(\mathbb{E}X)^2 = \frac{1-p}{\mathbb{E}X} \to 0 \tag{0.7}$$

when $n \to \infty$ by applying Second Moment Method
$\frac{1}{n^2}$ is indeed our threshold for $\mathbb{G}_{n,m}$

Sub-Critical Stage - Forests?

Theorem

If $m \ll n$, then $\mathbb{G}_m$ is a forest w.h.p..

Let $m = n/\omega$ and $N = \binom{n}{2}$. Then $p = m/N \leq 3/(\omega n)$. X as the number of cycles in $\mathbb{G}_{n,p}$.

$$\mathbb{E}X = \sum_{k=3}^{n} \binom{n}{k} \frac{(k-1)!}{2} p^k$$

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$$\leq \sum_{k=3}^{n} \frac{n^k}{k!} \frac{(k-1)!}{2} p^k$$

$$\leq \sum_{k=3}^{n} \frac{n^k}{2k} \frac{3^k}{(\omega n)^k} p^k = O(\omega^{-3}) \to 0$$

Forests? Continued

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$$\mathbb{P}(\mathbb{G}_{m} \text{ is a forest}) \to 1 \text{ as } n \to \infty \tag{0.9}$$

Applications of Random Graph

Can be used for finding bounds in fields like Ramsey Theory.
Networking of Graphs.
When special property of graphs occur i.e connectivity, perfect matching, hamiltonian cycle.
Read more

1. Frieze-Karonski Random Graphs
2. Bollobas Random Graphs
3. Hofstad Random Graphs