# Bertrand's Postulate

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# Introduction

- Bertrand's Postulate is an important statement in probability theory and mathematical statistics.
- It is named after the French mathematician Joseph Bertrand, who first formulated it in 1845.

# Formulation of the Postulate

•Bertrand's Postulate states that for any positive number n, there always exists at least one proper fraction $p/q$, where p and q are coprime integers, such that $1/n < p/q < 1/2$.

# Examples

Example 1: $n = 13$

•We want to find a proper fraction $p/q$such that $1/13 < p/q < 1/2$.

•Let's take $p = 3$ and $q = 8$. Then, $3/8 = 0.375$, which satisfies the inequality $1/13 < 3/8 < 1/2$.

•This demonstrates that for $n = 13$, there exists a proper fraction that fulfills Bertrand's Postulate.

Example 2: $n = 17$

•Again, we seek a proper fraction $p/q$ such that $1/17 < p/q < 1/2$.

•Taking $p = 5$ and $q = 12$, we have $5/12 = 0.4167$, which satisfies the inequality $1/17 < 5/12 < 1/2$.

•Thus, for$n = 17$, we can find a proper fraction that satisfies Bertrand's Postulate.

# Proof of Bertrand's Postulate

Consider an arbitrary positive integer$n$. We want to show that there exists at least one proper fraction $p/q$, where p and q are coprime integers, such that $1/n < p/q < 1/2$. Let's assume that there are no such fractions for a given $n$, and we will arrive at a contradiction. We consider the interval

$$[1/n, 1/2]$$

and divide it into two equal parts:

$$[1/n, 1/(2n)]$$

and

$$[1/(2n), 1/2]$$

. For any fraction $p/q$ within the interval

$$[1/n, 1/(2n)]$$

# Proof of Bertrand's Postulate

we can multiply both the numerator and the denominator by $2n$ to obtain an equivalent fraction within the interval

$$[2, q/n]$$

.

This contradiction implies that our assumption in step 3 is false. Hence, there must exist at least one proper fraction $p/q$, where $p$ and $q$ are coprime integers, such that $1/n < p/q < 1/2$for any positive integer$n$. This completes the proof of Bertrand's Postulate. Pál Turán's proof provides a rigorous demonstration of the existence of suitable fractions within the specified range for any positive integer $n$.

Since we assume that there are no suitable fractions within the interval

$$[1/n, 1/2]$$

, it implies that there are no fractions within the interval

$$[2, q/n]$$

. Now, let's consider the number of positive integers $k$ within the interval

$$[2, n]$$

such that there are no fractions p/q with denominator $q \leq k$ satisfying $1/k < p/q < 1/2$. By observing the intervals

$$[1/2, 1]$$

,

$$[1/3, 1/2]$$

,

# Proof of Bertrand's Postulate

$$[1/4, 1/3]$$

, ...,

$$[1/n, 1/(n-1)]$$

, we can conclude that the number of such integers k is less than or equal to the number of primes within the interval

$$[n/2, n]$$

. According to the Prime Number Theorem, the number of primes within the interval $[n/2, n]$ is approximately $n/(logn)$, as $n$ approaches infinity.

# Proof of Bertrand's Postulate

Combining the observations from steps 6, 7, and 9, we see that the number of positive integers k within the interval

$$[2, n]$$

for which there are no suitable fractions is less than or equal to $n/(logn)$. However, as n approaches infinity, the number of positive integers within the interval

$$[2, n]$$

is $n - 1$, which is greater than $n/(logn)$ for sufficiently large values of $n$. Hence, there must exist at least one proper fraction $p/q$, where $p$ and $q$ are coprime integers, such that $1/n < p/q < 1/2$ for any positive integer n. This completes the proof of Bertrand's Postulate. Pál Turán's proof provides a rigorous demonstration of the existence of suitable fractions within the specified range for any positive integer $n$.

# Significance of the Postulate

•Bertrand's Postulate is of significant importance in combinatorics, probability theory, and other areas of mathematics.
•It provides us with information about the distribution of prime numbers and helps in solving various problems related to probability.

# Conclusion

•Bertrand's Postulate speaks to the existence of proper fractions that satisfy a specific inequality.
•It has wide applications in various branches of mathematics and aids in better understanding the distribution of prime numbers.

Thank you for your attention