Integer Partitions and Applications to Number Theory

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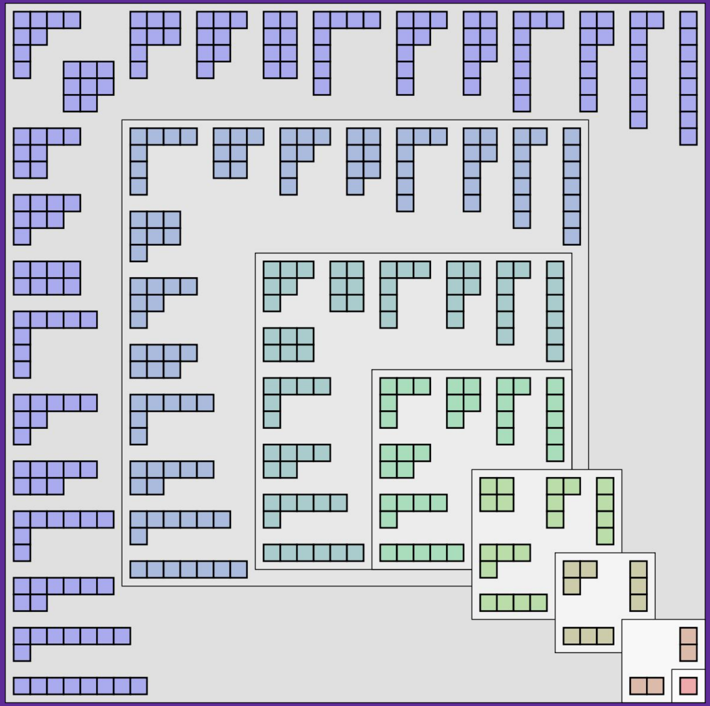
July 2023

Introduction

What are integer partitions?



Representations of Integer Partitions - Notation

Standard Notation

$$1+1+1+1+1+1+2+2+3+4+4+4+4+9=38$$

$$a_1 + a_2 + a_3 + \cdots + a_k$$

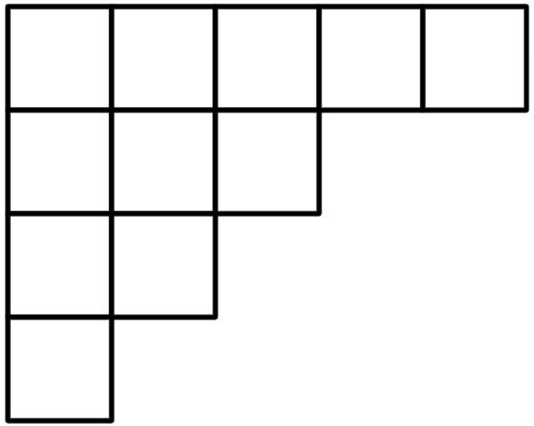
Multiplicity Notation

$$1^6 2^2 3^1 4^4 9^1$$

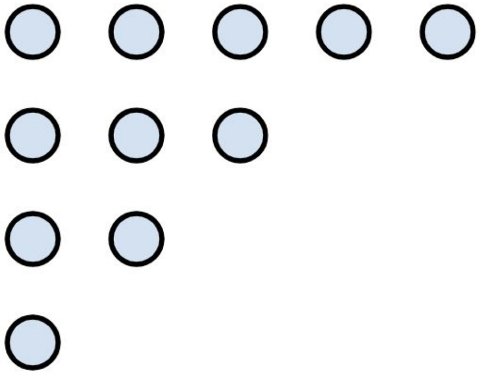
$${a_1}^{p_1} + {a_2}^{p_2} + {a_3}^{p_3} + \cdots + {a_k}^{p_k}$$

Representations of Integer Partitions - Diagrams

Ferrers Diagram

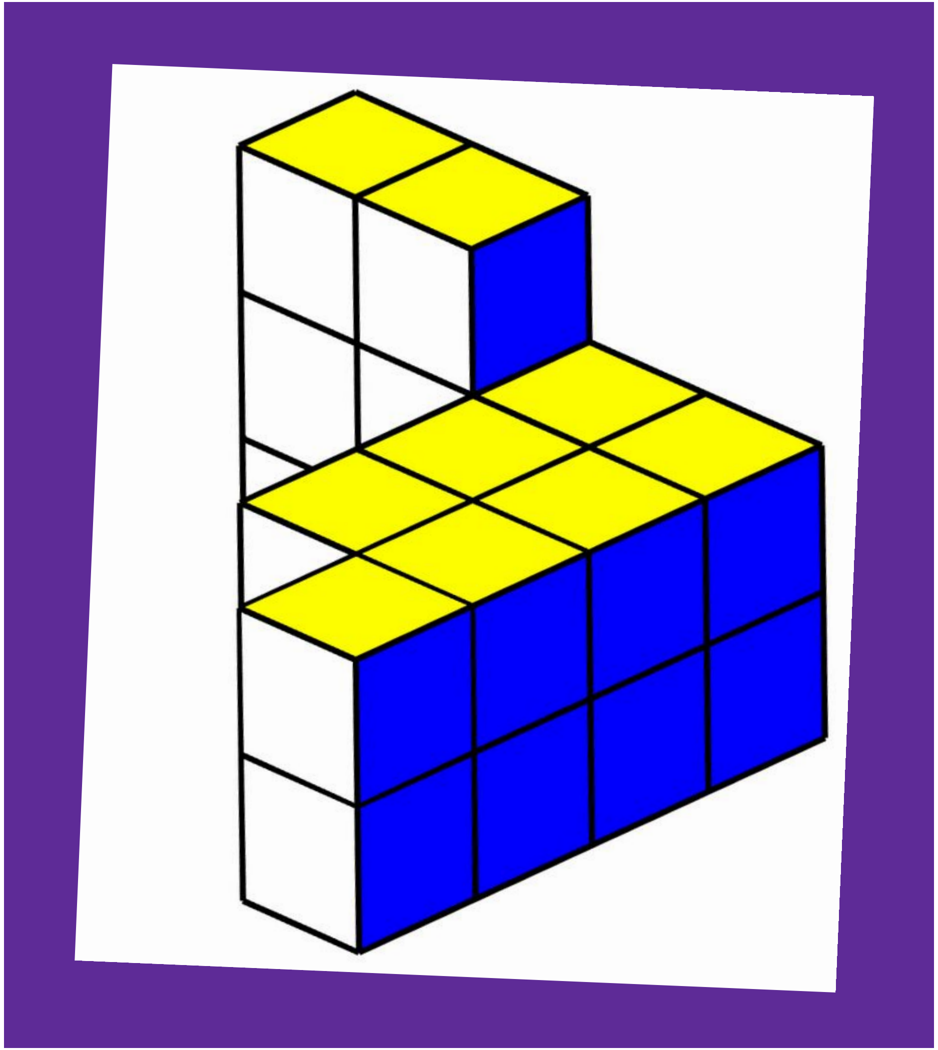


Young Diagram



Special Partitions

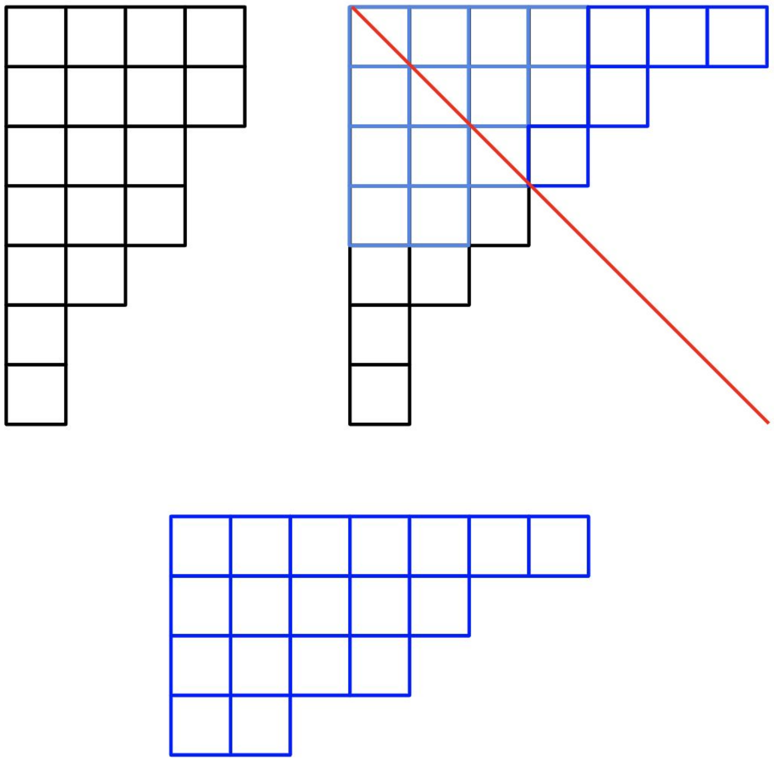
Conjugate Partitions
Restricted Partitions



Conjugate Partitions

4 + 4 + 3 + 3 + 2 + 1 + 1 = 18

7 + 5 + 4 + 2 = 18

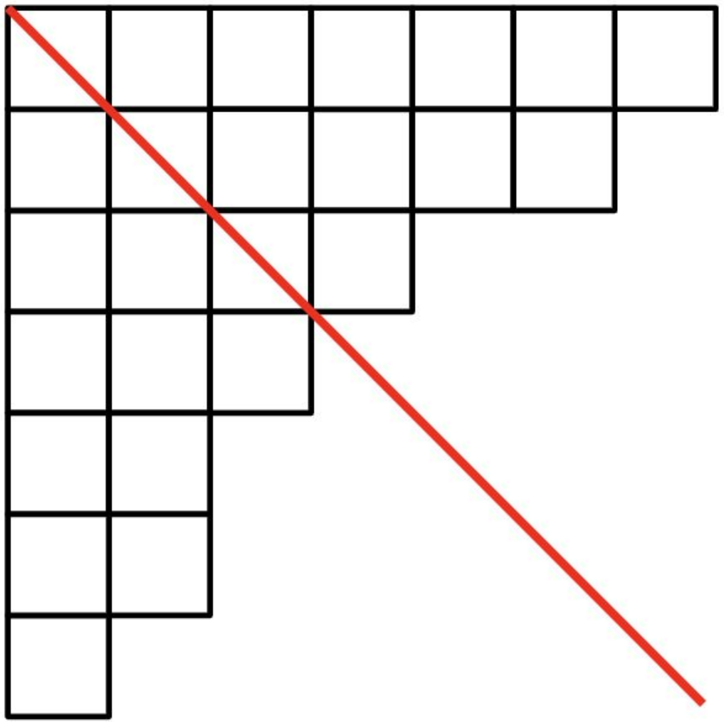


Self-Conjugate Partitions

7 + 6 + 4 + 3 + 2 + 2 + 1 = 25

7 + 6 + 4 + 3 + 2 + 2 + 1 = 25

The partition and its conjugate are congruent.



Distinct Parts Partitions

While the number 10 has 42 total partitions, it only has 10 distinct parts partitions:

$$\begin{aligned}
10 &= 10, \\
&= 9 + 1, \\
&= 8 + 2, \\
&= 7 + 3, \\
&= 7 + 2 + 1, \\
&= 6 + 4, \\
&= 6 + 3 + 1, \\
&= 5 + 4 + 1, \\
&= 5 + 3 + 2, \\
&= 4 + 3 + 2 + 1.
\end{aligned}$$

Odd Parts Partitions

$$
\begin{aligned}
10 &= 9 + 1, \\
&= 7 + 3, \\
&= 7 + 1 + 1 + 1, \\
&= 5 + 5, \\
&= 5 + 3 + 1 + 1, \\
&= 5 + 1 + 1 + 1 + 1 + 1, \\
&= 3 + 3 + 3 + 1, \\
&= 3 + 3 + 1 + 1 + 1 + 1, \\
&= 3 + 1 + 1 + 1 + 1 + 1 + 1 + 1, \\
&= 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1.
\end{aligned}
$$

Glaisher's Theorem

The number of distinct parts partitions and odd parts partitions of any positive integer will always be the same.

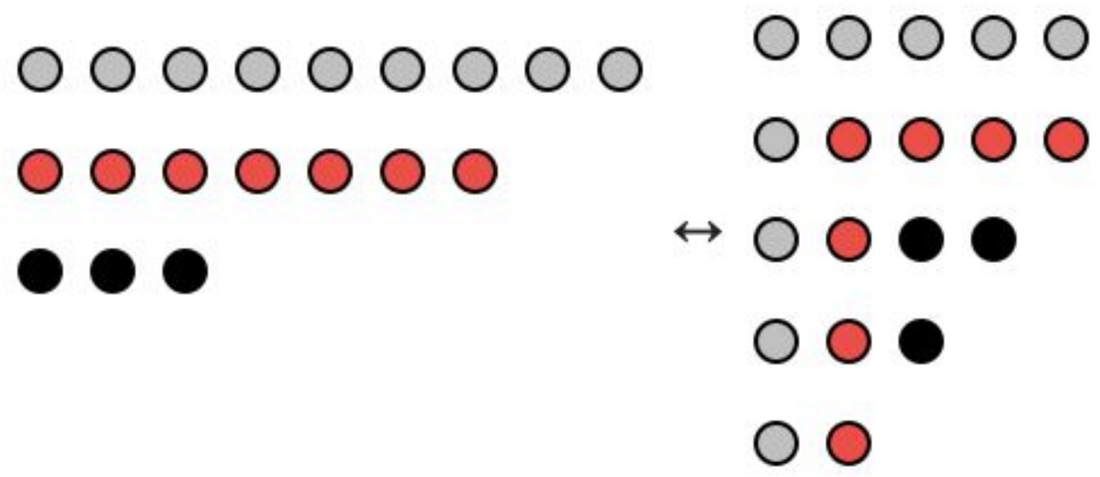
$$
\begin{aligned}
10 &= 10, & &= 9+1, \\
&= 9+1, & &= 7+3, \\
&= 8+2, & &= 7+1+1+1, \\
&= 7+3, & &= 5+5, \\
&= 7+2+1, & &= 5+3+1+1, \\
&= 6+4, & &= 5+1+1+1+1+1, \\
&= 6+3+1, & &= 3+3+3+1, \\
&= 5+4+1, & &= 3+3+1+1+1+1, \\
&= 5+3+2, & &= 3+1+1+1+1+1+1+1, \\
&= 4+3+2+1. & &= 1+1+1+1+1+1+1+1+1+1.
\end{aligned}
$$

Distinct Odd Parts Partitions

The distinct odd parts partitions of 20:

$$
\begin{aligned}
20 &= 19 + 1, \\
&= 17 + 3, \\
&= 15 + 5, \\
&= 13 + 7, \\
&= 11 + 9, \\
&= 11 + 5 + 3 + 1.
\end{aligned}
$$

Bijection Between Dist. Odd and Self-Conjugate

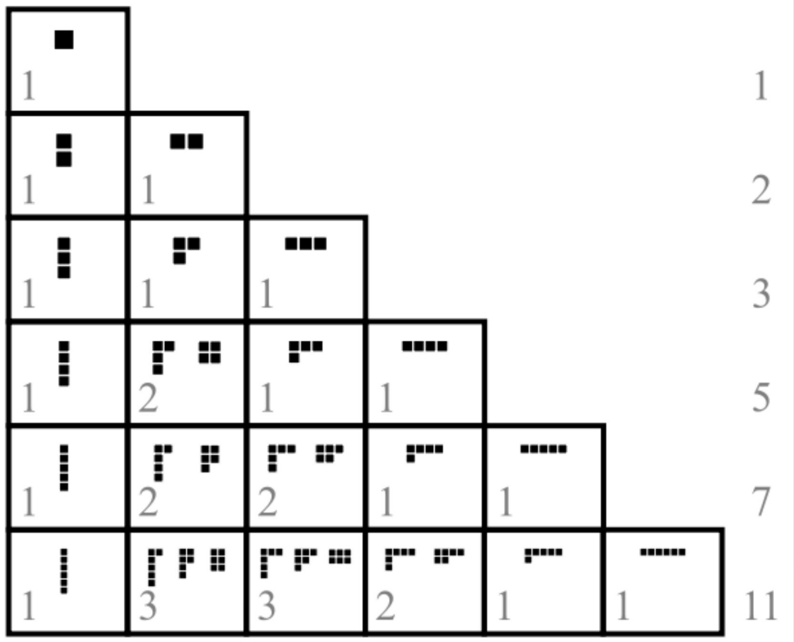


9 + 7 + 3 = 5 + 5 + 4 + 3 + 2

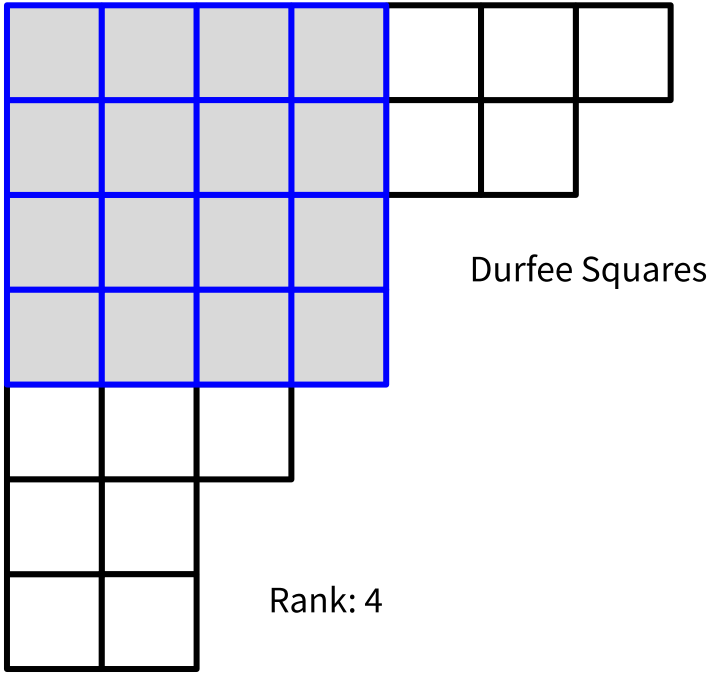
Dist. odd self-conjugate

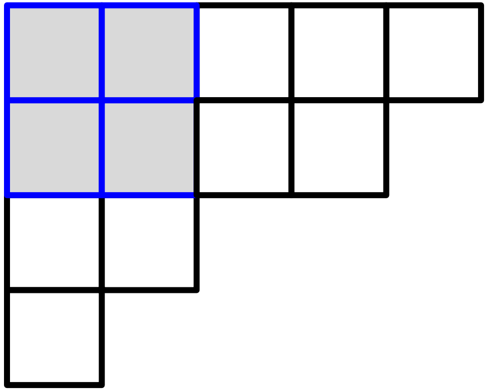
Qualities of Partitions

Partition Classification
The Partition Function
Congruences of Partitions



Rank of a Partition





The Partition Function

The partition function $p(n)$ is a function that returns the number of partitions of a positive integer n. For example, $p(5) = 7$ and $p(10) = 42$. Although no closed-form expression for the partition function exists, it turns out to have a relatively simple generating function. Below is the generating function for the partition function $p(n)$:

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{j=1}^{\infty} \sum_{i=0}^{\infty} q^{ji} = \prod_{j=1}^{\infty} (1 - q^j)^{-1}$$

Ramanujan's Congruences

$$p(5k+4) \equiv 0 \pmod{5},$$
$$p(7k+5) \equiv 0 \pmod{7},$$
$$p(11k+6) \equiv 0 \pmod{11}.$$

Srinivasa Ramanujan

$$p(17303k+237) \equiv 0 \pmod{13},$$
$$p(206839k+2623) \equiv 0 \pmod{17},$$
$$p(1977147619k+815655) \equiv 0 \pmod{19}.$$

A.O.L. Atkin

And so on…

Others

Thank you for watching!

	Partition of the number K																																			
	K approaches infinity		For finite K, these are stage values																																	
Totals	Stage X= (K - #spaces + 1)	K is a finite number	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34
1	1	0	1																																	
1	2	1	1																																	
2	3	2	1	1																																
3	4	3	1	1	1																															
5	5	4	1	1	2	1																														
7	6	5	1	1	2	2	1																													
11	7	6	1	1	2	3	3	1																												
15	8	7	1	1	2	3	4	3	1																											
22	9	8	1	1	2	3	5	5	4	1																										
30	10	9	1	1	2	3	5	6	7	4	1																									
42	11	10	1	1	2	3	5	7	9	8	5	1																								
56	12	11	1	1	2	3	5	7	10	11	10	5	1																							
77	13	12	1	1	2	3	5	7	11	13	15	12	6	1																						
101	14	13	1	1	2	3	5	7	11	14	18	18	14	6	1																					
135	15	14	1	1	2	3	5	7	11	15	20	23	23	16	7	1																				
176	16	15	1	1	2	3	5	7	11	15	21	26	30	27	19	7	1																			
231	17	16	1	1	2	3	5	7	11	15	22	28	35	37	34	21	8	1																		
297	18	17	1	1	2	3	5	7	11	15	22	29	38	44	47	39	24	8	1																	
385	19	18	1	1	2	3	5	7	11	15	22	30	40	49	58	57	47	27	9	1																
490	20	19	1	1	2	3	5	7	11	15	22	30	41	52	65	71	70	54	30	9	1															
627	21	20	1	1	2	3	5	7	11	15	22	30	42	54	70	82	90	84	64	33	10	1														
792	22	21	1	1	2	3	5	7	11	15	22	30	42	55	73	89	105	110	101	72	37	10	1													
1002	23	22	1	1	2	3	5	7	11	15	22	30	42	56	75	94	116	131	136	119	84	40	11	1												
1255	24	23	1	1	2	3	5	7	11	15	22	30	42	56	76	97	123	146	164	163	141	94	44	11	1											
1575	25	24	1	1	2	3	5	7	11	15	22	30	42	56	77	99	128	157	186	201	199	164	108	48	12	1										
1958	26	25	1	1	2	3	5	7	11	15	22	30	42	56	77	100	131	164	201	230	248	235	192	120	52	12	1									
2436	27	26	1	1	2	3	5	7	11	15	22	30	42	56	77	101	133	169	212	252	288	300	282	221	136	56	13	1								
3010	28	27	1	1	2	3	5	7	11	15	22	30	42	56	77	101	134	172	219	267	318	352	364	331	255	150	61	13	1							
3718	29	28	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	174	224	278	340	393	434	436	391	291	169	65	14	1						
4565	30	29	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	175	227	285	355	423	488	525	522	454	333	185	70	14	1					
5604	31	30	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	229	290	366	445	530	598	638	618	532	377	206	75	15	1				
6842	32	31	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	230	293	373	460	560	653	732	764	733	612	427	225	80	15	1			
8349	33	32	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	231	295	378	471	582	695	807	887	919	860	709	480	249	85	16	1		
10143	34	33	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	231	296	381	478	597	725	863	984	1076	1090	1009	811	540	270	91	16	1	
12310	35	34	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	231	297	383	483	608	747	905	1060	1204	1291	1297	1175	931	603	297	96	17	1