Integer Partitions and Applications to Number Theory

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July 2023

Abstract

Integer partitions are a fascinating topic in mathematics that deals with expressing a positive integer as a sum of smaller positive integers. The concept of partitioning has been studied for centuries and has applications in various fields, including number theory, combinatorics, and computer science. The number of distinct partitions of a given integer is a well-studied problem, and it grows rapidly with the size of the integer. Understanding the properties and patterns of integer partitions provides valuable insights into the nature of numbers and the intricate connections between different areas of mathematics.

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1 Introduction

Among the simplest of the countless operations used in mathematics is addition: combining numbers into a new, larger number that represents its constituent parts. This is the core idea of partitioning integers: splitting positive integers into smaller positive integers of which they are composed. As this allows to show the ways in which you can build a number through the summation of smaller numbers, this is a very useful and practical approach to solving many problems in combinatorics. Furthermore, integer partitions have various applications to number theory and other aspects of discrete mathematics.

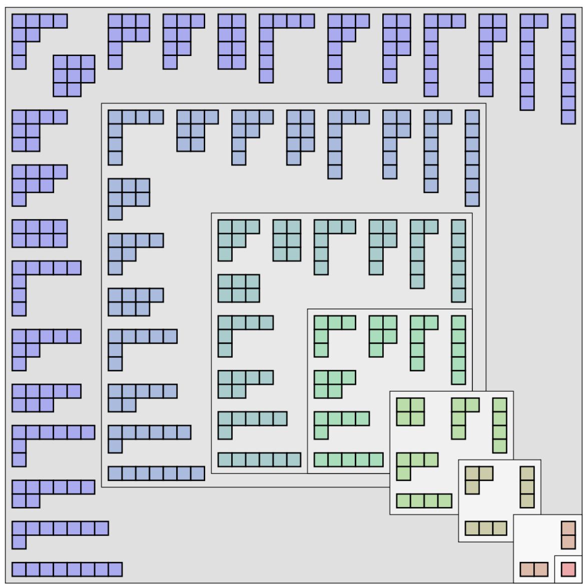

Figure 1: Image courtesy of Wikimedia.

Let's describe partitions mathematically as well. A partition of an integer n is defined as any ordered group of positive integers whose sum is n, or any non-increasing sequence $a_1, a_2, a_3, \cdots, a_k$ such that

$$\sum_{k=1}^{k} a_j = n$$

For example, the number $n = 4$ has five partitions: 4, $3+1$, $2+2$, $2+1+1$, and $1+1+1+1$.

Keep in mind that this means that any two partitions are the same if they contain all of the same elements, regardless of order. For example, the $2+4+4$ partition of 10 is the same as the $4+2+4$ partition of

10. Partitions are generally written in descending order, so you will most likely see this particular partition written as $4 + 4 + 2$.

Because the order of elements in a partition is not taken into account, the simpler notions of combinatorics are insufficient to provide us with a way of accurately determining the number of partitions for any given positive integer n. This, of course, leads to the partition function $p(n)$ that returns the number of partitions of an integer for a given positive integer n, which will be covered later on in this paper.

This paper will explore integer partitions in depth and discuss its ties to numerous fields of mathematics. We will go over the diagrams associated with partitions, several interesting theorems relating to the diagrams, the partition function $p(n)$, Ramanujan's congruences, and more.

Acknowledgements

I would like to extend my heartfelt gratitude to Herbert S. Wilf for his invaluable contributions through his explanatory text. Special thanks to Simon Rubinstein-Salzedo for providing the opportunity to write this paper and for invaluable guidance throughout the process. Finally, sincere appreciation to Lucas Perry for his exceptional support as an outstanding TA, adding tremendous value to my learning experience.

2 Representations of Integer Partitions

Individual integer partitions can be visualized in a number of ways diagrammatically. The following diagrams help make a particular partition more clear not only of the partition itself but also of some interesting properties of partitions, as will be shown later.

2.1 The Ferrers Diagram

To represent a partition of an integer, we must first know the important parts of integer partitions: the integer itself, the number of partitions it is split up into, and the values of each of those partitions. Since order does not matter for partitions, they'll need some set order as well, to distinguish them. To cover all of these, we'll be arranging a series of circles, or dots, in a way that represents the full integer followed by each of its components in decreasing order. For example, the $5 + 3 + 2 + 1$ partition of $n = 11$:

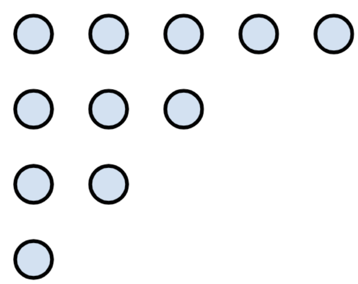

Figure 2: The Ferrers diagram, displaying the partition $5+3+2+1 = 11$.

The four rows have $5, 3, 2$, and 1 dot each, representing each of the partitions. The total, n, is represented by all of the dots as a group. The number of rows is the number of partitions, and the number of columns is the largest partition in the non-increasing sequence.

2.2 The Young Diagram

Many people actually consider the Young and Ferrers diagrams to be one and the same. The reason for this is because the Young diagram *is* the exact same thing as the Ferrers diagram, but instead of circles or dots, it uses adjacent grid squares, as demonstrated below:

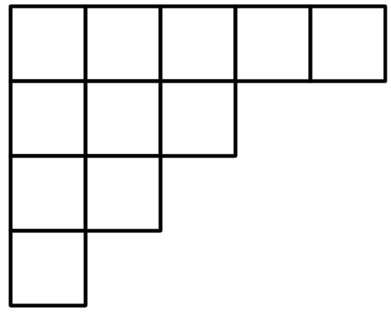

Figure 3: A Young diagram of the partition $5 + 3 + 2 + 1 = 11$, same as the Ferrers diagram above.

As you may be able to tell, this diagram is in every way identical to the previous diagram except for the shape of the individual parts. It shows the same four rows of $5, 3, 2,$ and 1, but rather than taking the form of dots, they are squares. This shows us that we will have the same information and be able to do the same things regardless of how we represent it diagrammatically, which leaves the choice of diagram as more of a simple choice of preference, and changing diagram style will have no effect.

2.3 Multiplicity Notation

Integer partitions are normally notated as $a_1 + a_2 + a_3 + \cdots + a_k$ for a partition of n with k elements, where the non-increasing sequence a represents the partition and $a_i \geq a_{i+1}$ for $1 \leq i \leq k - 1$. However, another notation exists for partitions called multiplicity notation, and it is often used for compactness when there is a large number of partitions, or when a partition has a large number of elements. Multiplicity notation is shown as ${a_1}^{p_1} + {a_2}^{p_2} + {a_3}^{p_3} + \cdots + {a_k}^{p_k}$, where the sequence p represents the respective number of times each element appears in the partition. For example, the partition $1+1+1+1+1+1+2+2+3+4+4+4+4+9$ of 38 can be represented in multiplicity notation as $1^6 2^2 3^1 4^4 9^1$. Note that this does not mean that these numbers are multiplied, this is simply another way of representing a partition of an integer.

This paper will largely use the standard notation for partitions, but it also may use multiplicity notation interchangeably. To avoid confusion with actual multiplication, I have taken the liberty of mentioning the notation of any partitions I write in multiplicity notation.

3 Using Partitions Diagrams

For this section, we will be arbitrarily using Young Diagrams to represent partitions, as either diagram would work perfectly well.

3.1 Conjugate partitions

In integer partitions, the conjugate of any partition is defined as a non-increasing sequence of the number of elements in the partition that are k or greater for $k \in \mathbb{Z}^+$ and k is no greater than the largest part in the partition. Diagrammatically, this can be represented by reflecting the partition diagram across its diagonal, like so:

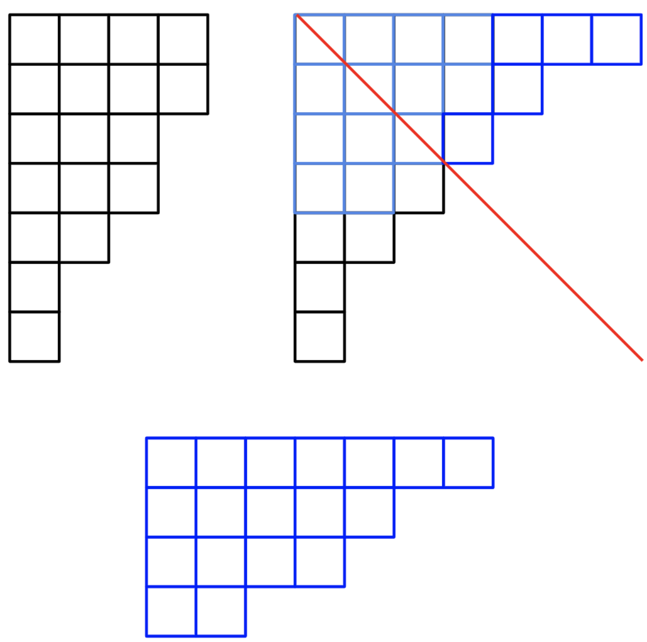

Figure 4: A partition and its conjugate.

In the above figure, the partition $4 + 4 + 3 + 3 + 2 + 1 + 1 = 18$ is "flipped", turning it into its conjugate partition, $7 + 5 + 4 + 2 = 18$. Note that just as transformations do not affect the area of a shape, a partition and its conjugate are still partitions of the same number n. As discussed earlier, each of the numbers in the newly created conjugate partition refers to the number of elements in the original partition that are greater than or equal to a number between 1 and the size of the largest element in the original partition. For $k = 1$, for example, we have 7 total elements that are 1 or greater. Then, there are 5 elements greater than or equal to $k = 2$, 4 elements greater than or equal to $k = 3$, and finally, two elements that are equal to $k = 4$.

Given a partition $A = a_1, \ldots, a_k$. Let $M = a_1$. Then the conjugate $B := \mathrm{conj}(A)$ is defined as the non-increasing sequence $b_1, \ldots, b_M$ where as $b_j = \#\{p : a_p \geq j\}$ for $1 \leq j \leq M$, where $\#$ denotes cardinality of the finite set.

Because a conjugate's conjugate returns us to the original partition, this is also true the other way around from the conjugate back to the original.

3.2 Self-Conjugate Partitions

When a particular partition's conjugate is exactly the same as the original partition, it is called a self-conjugate partition. Since conjugate partitions are formed by a reflection of the partition diagram across its diagonal, self-conjugate partitions must be symmetrical along its diagonal as its axis. One such partition is shown below:

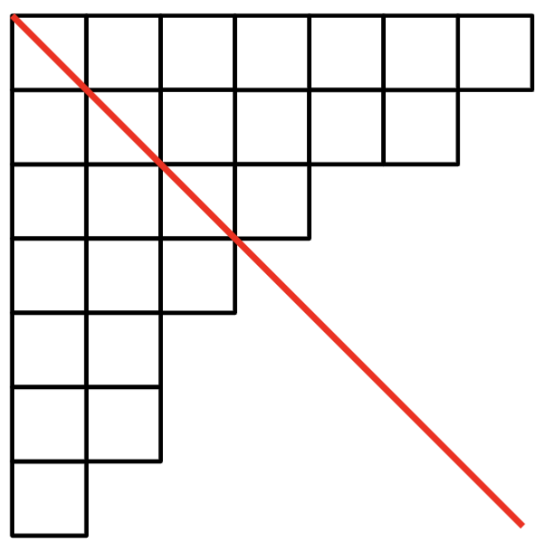

Figure 5: An example of a self-conjugate partition.

Because its conjugate partition just leads back to itself, the $7 + 6 + 4 + 3 + 2 + 2 + 1$ partition of 25 is a self-conjugate partition.

4 Restricted Partitions

In the fields of combinatorics and number theory, the study of partition families that fall within specific restrictions is a common area of focus. We will go over several such restrictions that are frequently examined. By imposing certain conditions on partitions, we aim to uncover unique patterns and properties within these restricted families. These restrictions may involve constraints on the sizes of the parts, the number of parts, or the arrangement of parts in a partition. By exploring these various restrictions, mathematicians gain a deeper understanding of the intricate relationships and structures that underlie the realm of integer partitions.

4.1 The Distinct Parts Partition

One example of a restricted partition is the distinct parts partition. These are the partitions with distinct elements; that is, no element appears twice in the same partition. These partitions are colloquially known as strict partitions. The $5 + 1 + 1$ partition of 6 is not a distinct parts partition, whereas the $7 + 4 + 2 + 1$ partition of 14 is.

While the number 10 has 42 total partitions, it only has 10 distinct parts partitions:

$$\begin{aligned}
10 &= 10, \\
&= 9 + 1, \\
&= 8 + 2, \\
&= 7 + 3, \\
&= 7 + 2 + 1, \\
&= 6 + 4, \\
&= 6 + 3 + 1, \\
&= 5 + 4 + 1, \\
&= 5 + 3 + 2, \\
&= 4 + 3 + 2 + 1.
\end{aligned}$$

4.2 The Odd Parts Partition

Another restricted partition is the odd parts partition. These partitions are, as you may have guessed, simply the partitions of an integer that contain only an odd number of parts. Below are the ten odd parts partitions of 10:

$$\begin{aligned}10 &= 9+1,\\ &= 7+3,\\ &= 7+1+1+1,\\ &= 5+5,\\ &= 5+3+1+1,\\ &= 5+1+1+1+1+1,\\ &= 3+3+3+1,\\ &= 3+3+1+1+1+1,\\ &= 3+1+1+1+1+1+1+1,\\ &= 1+1+1+1+1+1+1+1+1+1.\end{aligned}$$

4.3 Glaisher's Theorem

Did you notice that the number of odd parts partitions of $n = 10$ is equal to the number of distinct parts partitions? This is actually true of all positive n. In 1883, English mathematician and astronomer James Whitbread Lee Glaisher proved that the number of odd parts partitions and distinct parts partitions were equal regardless of n.

4.4 The Distinct Odd Parts Partition

The distinct odd parts partition is the intersection of the distinct parts partitions and the odd parts partitions, made up only of distinct odd parts. All positive integers except for 2 have at least one distinct odd parts partition. Here are the distinct odd partitions of 20, for example:

$$\begin{aligned}20 &= 19+1,\\ &= 17+3,\\ &= 15+5,\\ &= 13+7,\\ &= 11+9,\\ &= 11+5+3+1.\end{aligned}$$

The distinct odd parts partitions are very interesting because we can prove that they have a one-to-one correspondence with self-conjugate partitions, which were discussed in Section 3.2. Let us revisit self-conjugate partitions for a moment.

Any partition is self-conjugate if reflecting its Ferrers diagram across its diagonal does not change the partition at all. This means that on either side of the diagonal, the diagram must be a mirror image of itself: any detail that appears on one side must be apparent on the other.

5 Rank of a partition

Integer partitions are classified by their rank. This rank is determined by the side length of the Durfee Square of the partition, which is the largest square that fits in a Ferrers diagram representation of a partition starting from the top left. The rank can also be represented as a positive integer s where s is the largest number such that there are at least s elements of size s in the partition.

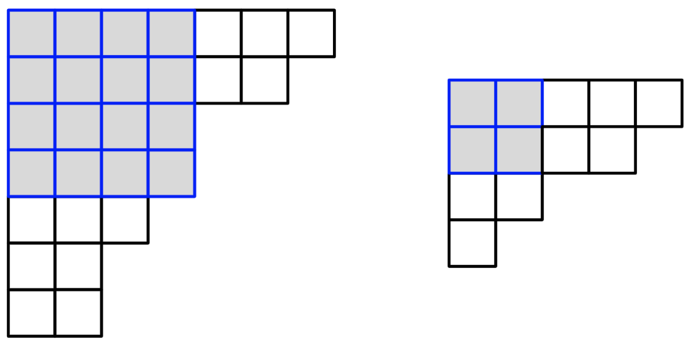

Figure 6: The Durfee squares of sizes 4 and 2 in partitions, respectively.

In the above figure, the $7+6+4+4+3+2+2$ partition of 28 has a Durfee square of side length 4 because there are four elements that are ≥ 4: $7, 6, 4$, and 4. Since this does not work for the next number, 5, this is the rank of our partition. Similarly, the $5+4+2+1$ partition of 12 to its right has a Durfee square of side length 2 because there are more than two elements larger than 2, but not three ore more elements larger than 3. Thus, it has a Durfee square of size 2 and thus an equal rank.

However, as it turns out, there is yet another definition of the rank of a partition. In 2005, mathematician and physicist Freeman Dyson suggested the notion of the "Rank of the Partition", which would be calculated as the number of elements subtracted from the largest element in the partition. Thus, the values for the rank of the partition in this definition can be negative, and range from $-(n-1)$ to $(n-1)$. This alternate method of classification is helpful for determining particular congruences relating to integer partitions. Below is a helpful illustration for calculating this rank using a Ferrers diagram.

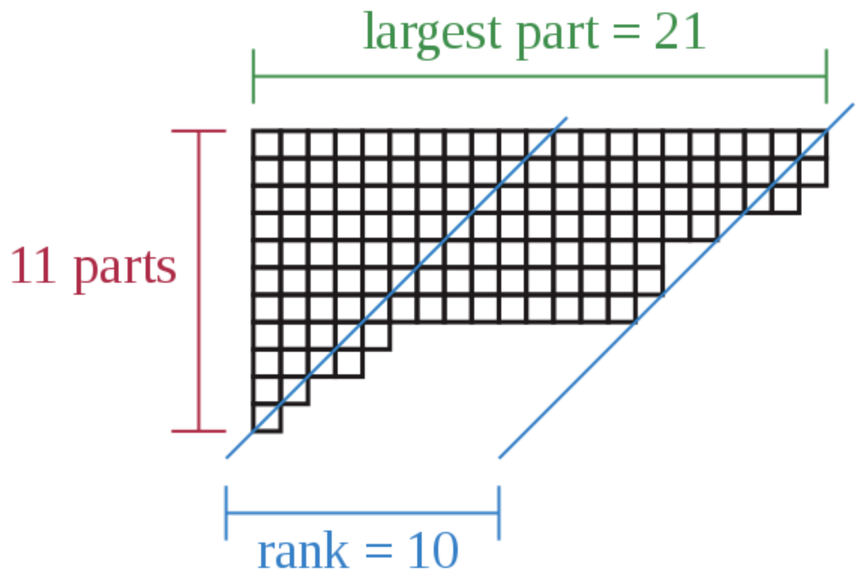

Figure 7: Image courtesy of Wikimedia.

6 The Partition Function

The partition function $p(n)$ is a function that returns the number of partitions of a positive integer n. For example, $p(5) = 7$ and $p(10) = 42$. Although no closed-form expression for the partition function exists, it turns out to have a relatively simple generating function. Below is the generating function for the partition function $p(n)$:

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{j=1}^{\infty}\sum_{i=0}^{\infty} q^{ji} = \prod_{j=1}^{\infty}(1-q^j)^{-1}$$

There is also the partition function $q(n)$ which produces the number of strict partitions, or distinct parts partitions, for a positive integer n. This function's generating function also turns out to be relatively simple:

$$\sum_{n=0}^{\infty} q(n)x^n = \prod_{k=1}^{\infty}(1+x^k) = \prod_{k=1}^{\infty}\frac{1}{1-x^{2k-1}}$$

Euler's pentagonal number theorem can be used to create the generating function for $p(n)$. It relates the product and series representations of the Euler function, shown below:

$$(1-x)(1-x^2)(1-x^3)\cdots = \prod_{n=1}^{\infty}(1-x^n) =$$

$$\sum_{k=-\infty}^{\infty} (-1)^k\, x^{k(3k-1)/2} = 1 + \sum_{k=1}^{\infty}(-1)^k \left(x^{\frac{k(3k+1)}{2}} + x^{\frac{k(3k-1)}{2}}\right) =$$

$$1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \ldots$$

[Wil00]

The exponents of the expression form a series of integers known as the pentagonal numbers, which as it turns out, helps us create a recurrence for calculating a formula for $p(n)$:

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + \cdots$$

or,

$$p(n) = \sum_{k \neq 0} (-1)^{k-1} p(n - g_k)$$

where g_k represents the $(k-1)$th pentagonal number.

6.1 Ramanujan's Congruences

In his 1919 paper, Indian mathematician Srinivasa Ramanujan published his discovery and proof of three congruences relating to integer partitions, and they are as follows:

$$p(5k+4) \equiv 0 \pmod{5}, \quad (1)$$
$$p(7k+5) \equiv 0 \pmod{7}, \quad (2)$$
$$p(11k+6) \equiv 0 \pmod{11}. \quad (3)$$

This means that:

1. If a number is 4 more than a multiple of 5, then the number of its partitions is a multiple of 5.

2. If a number is 5 more than a multiple of 7, then the number of its partitions is a multiple of 7.

3. If a number is 6 more than a multiple of 11, then the number of its partitions is a multiple of 11.

Unfortunately, however, this pattern does not continue, and further potential congruences such as $p(13k+7) \equiv 0 \pmod{13}$ have been proven to be false. However, much later, additional congruences for prime moduli in integer partitions were discovered, but they were far more complex and followed no distinguishable pattern.

In the 1960s, British mathematician Arthur Oliver Lonsdale Atkin (A.O.L. Atkin) discovered the following congruences:

$$p(17303k + 237) \equiv 0 \pmod{13}, \tag{4}$$
$$p(206839k + 2623) \equiv 0 \pmod{17}, \tag{5}$$
$$p(1977147619k + 815655) \equiv 0 \pmod{19}. \tag{6}$$

Further studies yielded proof of a much larger amount of similar congruences of integer partitions, and mathematicians such as R.L. Weaver in 2001 and F. Johansson in 2012 succeeded in creating effective algorithms for calculating another 22474608014 congruences.

References

[Wil00] Herbert S. Wilf. Lectures on integer partitions. 2000.