The Group Extension Problem

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July 6 2023

Group Decomposition

Definition 1.11 A **maximal normal subgroup** N of G is a proper normal subgroup N such that if $N \leq K$ and $K \trianglelefteq G$ then $K = N$ or $K = G$.

Definition 1.8 A **simple** group is a nontrivial group G such that the only normal subgroups of G are the trivial subgroups.

Theorem 1.12 A normal subgroup $N \trianglelefteq G$ is maximal iff G/N is simple.

Definition 1.13 Let G be a group. We call a series $A_1 \leq \cdots \leq A_n$ of subgroups of G such that $A_1 = \{e\}$ and $A_n = G$ a **composition series** for G if each term (other than A_n) is a maximal normal subgroup of the next. If you take A_{i+1}/A_i for each i you get a **decomposition** for G.

Theorem 1.9 Any two decompositions of any group G are the same up to reordering.

The Group Extension Problem

Definition 2.2 Take any two groups K and Q. A group G such that $K \trianglelefteq G$ and $G/K \cong Q$ is called an **extension** of K by Q. More presicely, if there's a normal subgroup $K' \trianglelefteq G$ that's isomorphic to K, and if $G/K' \cong Q$, then we say G is an extension of K by Q.
Definition 2.3 The **Group Extension Problem**, formulated by O. Hölder, is the problem of, for given groups K and Q, finding all extensions of K by Q.

Transversals

Definition 2.8 If G is an extension of K by Q, we call a function $l : Q \to G$ a **transversal** if for any coset $a \in G/K$, $l(a) \in a$.

Realizing Data

Definition 2.9 We call an ordered triple (Q, K, θ) **data** if Q is a group, K is an abelian group, and $\theta : Q \to \mathrm{Aut}(K)$ is a homomorphism. We say that a group G **realizes** this data if G is an extension of K by Q and, for every transversal $l : Q \to G$,

$$\theta_x(a) = \theta(x)[a] = l(x) + a - l(x).$$

(Note that we will be using additive notation for the operations in G and K, breaking convention. We'll still use multiplicative notation for Q, though.) We also denote $\theta_x(a)$ by xa.

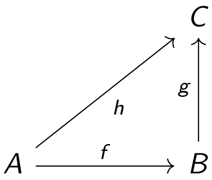
Factor Sets

Definition 2.12 If G realizes data (Q, K, θ) and $l : Q \to G$ is a transversal such that $l(1) = 0$, then the **factor set** $f : Q \times Q \to K$ (also called a **cocycle**) arising from the transversal l is defined so that $f(x, y) = l(x) + l(y) - l(xy)$. f essentially measures how 'far away' l is from being a homomorphism.

Commutative Diagrams

Definition 2.16 Consider any diagram of functions between sets. We say that the diagram **commutes** if, for any two fixed sets A and C on the map, the composition of any chain of functions along the diagram from A to C is always the same.

Example 2.17 The diagram below commutes iff $h = g \circ f$:



Equivelant Extensions

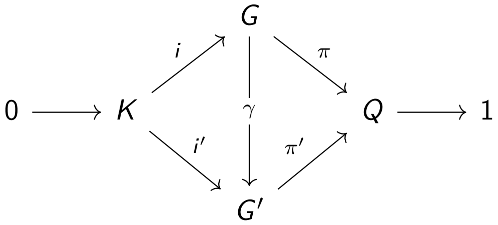
Definition 2.18 Take any homomorphism $\phi : G \to H$. We denote $\ker\phi$, called the **kernel** of ϕ to be the set of all $x \in G$ such that $\phi(x) = e$. We denote $\phi(G)$, or the **image** of ϕ to be the set of all $x \in H$ such that there's an $a \in G$ so that $\phi(a) = x$. Notice that $\ker\phi \trianglelefteq G$ and $\phi(G) \le H$.

Definition 2.19 We call a chain of homomorphisms between groups an **exact sequence** if the image of one is the kernel of the next, and we call it a **short exact sequence** if it's of the form

$$0 \longrightarrow K \xrightarrow{\ i\ } G \xrightarrow{\ \pi\ } Q \longrightarrow 1$$

The existance of such a sequence (for K, Q, and G) is an alternative way of saying that G is an extension of K by Q.

Definition 2.20 We say that two extenions G and G' are **equivalent** if there is an isomorphism $\gamma : G \to G'$ such that the following diagram commutes:



γ is called an **isomorphism of extensions**.

Coboundaries

Theorem 2.21 For any two factor sets f and f' arising from transversals of the same group extension G realizing data (Q, K, θ), there's a function $h : Q \to K$ with $h(1) = 0$ such that

$$f'(x, y) - f(x, y) = xh(y) - h(xy) + h(x).$$

Definition 2.22 We call $g : Q \times Q \to K$ a **coboundary** if there's a function $h : Q \to K$ with $h(1) = 0$ such that

$$g(x, y) = xh(y) - h(xy) + h(x).$$

Theorem 2.23 Let G and G' be extensions of K by Q. G and G' are equivalent iff they realize the same data (Q, K, θ) and there are factor sets f of G and f' of G' so that $f - f'$ is a coboundary.

The 2nd Cohomology Group

Definition 2.24 We denote $Z^2(Q, K, \theta)$ to be the set of all factor sets, and $B^2(Q, K, \theta)$ to be the set of all coboundaries.
It turns out that $Z^2(Q, K, \theta)$ forms an abelian group under $+$, where $(f+g)(x, y) = f(x, y) + g(x, y)$. $B^2(Q, K, \theta)$ forms a subgroup.
Definition 2.25 The **second cohomology group**, denoted $H^2(Q, K, \theta)$, is defined to be the quotient group $Z^2(Q, K, \theta)/B^2(Q, K, \theta)$. This is well-defined since all subgroups of abelian groups are normal.

Bijection between Equivalent Extensions and H_2

Theorem 2.26 Let E be the set of equivalence classes of extensions G realizing data (Q, K, θ). If you define $\phi : H^2(Q, K, \theta) \to E$ so that $\phi(f + B^2) = G_f$, then ϕ is a bijection.

Proof. First of all, we need to check well-definedness since it could depend on the choice of representative f. We know that if $g \in f + B^2(Q, K, \theta)$ then f and g differ by a coboundary, so since f and g are factor sets of G_f and G_g respectively, we know that G_f and G_g are equivalent.

Bijection between Equivalent Extensions and H_2

Conversely, if G_f and G_g are equivalent, then there are factor sets f' of G_f and g' of G_g that are in the same coset. But by Theorem (same group differ by a coboundary, will be numbered later) we know that g' and g lie in the same coset, and so do f and f', so indeed $g \in f + B^2(Q, K, \theta)$. This shows that ϕ is well-defined and injective. Surjectivity is a theorem in the paper, so then ϕ is a bijection.$\square$

So not only have we found out that G_f and G_g are equivalent when f and g differ by a coboundary, we can create a group structure on them where $G_f + G_g = G_{f+g}$, making ϕ an isomorphism!

Conclusion

To conclude, we split extensions up based on realization of data, defined what it means for them to be equivalent, and found a neat correspondence between equivalent extensions (realizing data) and the second cohomology group.