The Brachistochrone Problem

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Linear Spaces

Definition

A **normed vector space** or **normed linear space** is a vector space with a norm. More specifically, it represents a vector space $\mathcal{R}$ over some field F with a norm $\|\cdot\|$.

Normed Vector Space Axioms

Vector space axioms:

1. $u + (v + w) + (u + v) + w$;
2. $u + v = v + u$;
3. $\exists x \in V$ where $v + x = v$ (x is called the *zero vector*);
4. $\exists x \in V$ where $v + x = 0$ (x is called the *additive inverse*);
5. $a(bv) = ab(v)$;
6. $1v = v$;
7. $a(u + v) = au + av$;
8. $(a + b)v = av + bv$.

Norm axioms:

1. $f(v) = 0$ if and only if $v = 0$;
2. $f(a \cdot v) = a \cdot f(v)$;
3. $f(x + y) \leq f(x) + f(y)$.

What is a functional?

Definition

A **functional** I is a type of function that maps a function in $\mathcal{R}$ to $\mathbb{R}$.

Functionals are a type of function that assigns a real number to each function within a class- essentially, they map functions to numbers. Typically they are notated something like

$$J[y] = \int_a^b F(x, y, y')\, dx,$$

where $y' = \frac{dy}{dx}$.

Stationary points

Definition

A **stationary point** y of a functional I is a function that is the extremal of I; that is, $I[y]$ produces a value that is a local maximum or minimum of the functional when compared to all $I[\bar{y}]$, where $\bar{y}$ represents an infinitesimal change in y that still lies within our family of curves.

The Euler-Lagrange Equation

Stationary functions can be found using the Euler-Lagrange equation. In essence, the equation is a necessary condition for a function to be stationary (however, it is not sufficient). Eventually, we end up with the equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

The Beltrami Identity

The Beltrami Identity is a special case of the Euler-Lagrange equation. This occurs when

$$\frac{\partial F}{\partial x} = 0$$

for our functional F across all x. The identity only uses computations and the chain rule to eventually get the equation

$$F - y' \frac{\partial F}{\partial y'} = C,$$

where C is a constant of integration.
This will be used in our proof of the brachistochrone problem.

The Brachistochrone Problem

The Brachistochrone problem was introduced in 1696 by Johann Bernoulli. He posed the question as:

"Given two points A and B in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at A and reaches B in the shortest time."

The Brachistochrone Problem

Our function to be integrated is

$$t_0 = \int_{P1}^{P2} \frac{dS}{v}.$$

Conservation of energy states $K_1 + P_1 = K_2 + P_2$, and

$$KE = \frac{1}{2} mV^2$$
$$PE = mgh.$$

Thus,

$$mgy = \frac{1}{2} mV^2$$
$$v = \sqrt{2gy}.$$

The Brachistochrone Problem

Using the Pythagorean Theorem for dS, we can rewrite

$$dS = \sqrt{dx^2 + dy^2}$$

$$= \sqrt{1 + y'^2}dx.$$

The Brachistochrone Problem

So our integral becomes

$$t_0 = \int_{P1}^{P2} \frac{\sqrt{1 + y'^2}}{\sqrt{2gy}}\, dx,$$

and the function within the integral is the function to be varied.

The Brachistochrone Problem

Notice how since

$$F(y, y') = \sqrt{\frac{1 + y'^2}{2gy}}$$

doesn't explicitly contain x, we can apply the Beltrami Identity.

The Brachistochrone Problem

Recall that the Beltrami Identity is

$$F - y' \frac{\partial F}{\partial y'} = C.$$

We first need to find the partial derivative of F with respect to y', which is

$$\frac{\partial F}{\partial y'} = y'(y'^2 + 1)^{-\frac{1}{2}}(2gy)^{-\frac{1}{2}}.$$

Now we can plug our values into the Beltrami to get

$$\frac{1}{2gC^2} = y\left(1 + \left(\frac{dy}{dx}\right)^2\right).$$

The Brachistochrone Problem

We can set the left side of our equation to equal k^2 and we end up with the differential equation

$$y\left(1+\left(\frac{dy}{dx}\right)^2\right)=k^2.$$

This turns into the parametric equations

$$x=\frac{1}{2}k^2\left(\theta-\sin\theta\right),$$

$$y=\frac{1}{2}k^2\left(1-\cos\theta\right),$$

which happens to be the parametric equations of a cycloid. This is left as an exercise for the reader.