CALCULUS OF VARIATIONS: THE BRACHISTOCHRONE CURVE PROBLEM

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1. Abstract

This paper provides a background to calculus of variations and eventually solves the brachistochrone curve problem. To solve this, this article proves the Euler-Lagrange equation, which is an equation derived in the 1750s and is a condition for stationary points. Then the Beltrami Identity is explained, which is an identity for the Euler-Lagrange equation when $\frac{\partial F}{\partial x} = 0$. It is primarily used to solve the brachistochrone curve problem. The brachistochrone curve problem is our final result and will be proved last. Two methods for proving this problem are explained to provide more depth. Lastly, some other variational calculus problems are briefly explained if the reader is interested in learning more.

2. Introduction

Calculus of variations is thought to have originated with Newton's minimal resistance problem in 1687. The problem aims to find the solid of revolution that experiences minimal resistance when moving at a constant velocity in the direction of the axis of the solid's revolution. In essence, the problem requires the functional

$$I = \int \frac{yy'^3}{1+y'^2}\, dx$$

to be minimized, where $y(x)$ represents the curve that creates our solid of revolution when rotated about the x-axis, and $y'(x) = \frac{dy}{dx}$. This problem can be solved using the Euler-Lagrange equation explained later on which will be briefly explained in a later section.

In 1696, almost a decade later, the brachistochrone curve problem was introduced by Johann Bernoulli. In a scientific journal, he posed the question simply as:

"Given two points A and B in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at A and reaches B in the shortest time."

Johann and his brother Jakob both attempted to solve it, and Jakob was able to derive it successfully. Johann produced an incorrect proof and instead published his brother's solution as his own nearly a year later, at which point it would be proven by others as well.

For six months after the journal was published, Johann received zero responses. He extended the deadline and eventually, Isaac Newton caught wind of the problem and attempted it himself. The next day he submitted his solution,

Date: June 2023.

which turned out to be correct, and in the end, three others were able to produce a solution: l'Hôpital, Leibniz, and Von Tschirnhaus.
This problem pioneered the way for others to develop the field, and Leonhard Euler was the first to do so, along with Joseph-Louis Lagrange. The two developed the pivotal Euler-Lagrange equation in the 1750s while solving the tautochrone problem, which explores a curve for which a particle on the curve will reach the bottom of the curve at the same time as it would if it had started at any other point along the curve. (Thus one can deduce that none of the five original solutions to the brachistochrone problem used the Euler-Lagrange equation; however, this paper highlights a proof of the problem that does use the equation.) Interestingly enough, however, the tautochrone problem was first solved in 1659 and later published in 1673 by Christiaan Huygens; however, Huygens provided a geometric proof and so this problem is not considered to be the first instance of a variational calculus problem, despite having been solved decades before the minimal resistance problem.
Many prominent mathematicians made contributions to calculus of variations later on, including Augustin-Louis Cauchy and Emmy Noether, and in the 20th century Marston Morse applied calculus of variations to a field now called Morse Theory.
This paper will be focusing on the earlier calculus of variations problems. Many problems in this field have a solution using the Euler-Lagrange equation, which will be introduced first. After this the Beltrami Identity, which is used to solve the brachistochrone, will be explained in detail. The brachistochrone curve problem will be proved after this. The Beltrami is a modification of the Euler-Lagrange equation; therefore, the solution provided here for the brachistochrone curve problem is not one of the five original solutions received by Bernoulli; however, it is subjectively easier to follow and understand. Lastly some other calculus of variations problems will be suggested for the reader to look into independently if they are interested in learning more.

3. Preliminaries

As the name suggests, calculus of variations uses variations of functions and functionals to find the extremals of functionals. To define a functional, we introduce normed vector spaces:

Definition 1. A **normed vector space** or **normed linear space** is a vector space with a norm. More specifically, it represents a vector space $\mathcal{R}$ over some field F with a norm $\|\cdot\|$. This will be explained more thoroughly later on in this section.

Normed vector spaces allow us to define concepts such as distance and continuity within our space. A vector space must obey the following axioms for all $u, v, w \in V$ and $a, b \in F$:

(1) $u + (v + w) + (u + v) + w$;

(2) $u + v = v + u$;

(3) $\exists x \in V$ where $v + x = v$ (x is called the *zero vector*);

(4) $\exists x \in V$ where $v + x = 0$ (x is called the *additive inverse*);

(5) $a(bv) = ab(v)$;

(6) $1v = v$;

(7) $a(u + v) = au + av$;

(8) $(a + b)v = av + bv$.

For our vector space to be normed, each $v \in \mathcal{R}$ must be assigned a norm. A norm is a function $f : \mathcal{R} \to \mathbb{R}$ that heeds the following axioms for all $v \in \mathcal{R}$ and $a \in F$:

(1) $f(v) = 0$ if and only if $v = 0$;

(2) $f(a \cdot v) = a \cdot f(v)$;

(3) $f(x + y) \leq f(x) + f(y)$.

A normed vector space is important because it allows us to assign a value such as distance or time to each vector space. This segues nicely into our definition for functionals:

Definition 2. A functional I is a type of function that maps a function in $\mathcal{R}$ to $\mathbb{R}$.

Simply put, functionals are a type of function that assigns a real number to each function within a class. For instance, if we consider the class of all functions that connect two points A and B, then a functional could be obtained by associating each function with the total length of the path. A typical functional is notated something like

$$J[y] = \int_a^b F(x, y, y')\, dx,$$

where $F \in \mathbb{C}^2$ (that is, F' and F'' exist, both of which are continuous) and is a function of three variables. Functionals essentially take functions and map them to numbers, much like a normed vector space.
Functions that are the extremals of the functional are called *stationary points* of the functional. More specifically:

Definition 3. (stationary point): A **stationary point** y of a functional I is a function that is the extremal of I; that is, $I[y]$ produces a value that is a local maximum or minimum of the functional when compared to all $I[\bar{y}]$, where $\bar{y}$ represents an infinitesimal change in y that still lies within our family of curves.

Stationary points can be found using the Euler-Lagrange equation, which is derived and explained later on.

4. The Euler-Lagrange Equation

We will now attempt to find an equation that must be true for a function that makes a functional stationary. Our result is known as the **Euler-Lagrange equation** and can be used to solve many other problems in calculus of variations. This includes the tautochrone problem, for which the Euler-Lagrange equation was originally developed, and the brachistochrone, which will be proved later on in this paper. Such problems can be solved without the Euler-Lagrange equation, but using the formula makes it significantly simpler.

Say we have two points $A, B \in \mathbb{R}^2$ where $A = (x_1, y_1)$ and $B = (x_2, y_2)$. Find a function y such that the following functional is stationary:

$$I = \int_{x_1}^{x_2} F(x, y, y')\, dx.$$

A and B are our fixed boundary points for this function, so it must also be the case that $y(x_1) = y_1$ and $y(x_2) = y_2$.
So suppose $y(x)$ makes I stationary and satisfies our boundary conditions. Let $\eta(x)$ be a function where $\eta(x_1) = \eta(x_2) = 0$. Then we define

$$\bar{y}(x) = y(x) + \epsilon\eta(x)$$

for some ϵ. Notice that because $\eta(x_1) = \eta(x_2) = 0$, $\bar{y}(x_1) = y_1$ and $\bar{y}(x_2) = y_2$ and so $\bar{y}(x)$ has the same bounds.
Because $\eta(x)$ is arbitrary, $\bar{y}(x)$ can represent any arbitrary function so long as it conforms to the previous restrictions set on $\eta(x)$ and $y(x)$. $\bar{y}(x)$ represents a family of curves, but we will attempt to find the specific curve $\bar{y}(x)$ that makes

$$I(\epsilon) = \int_{x_1}^{x_2} F(x, \bar{y}, \bar{y}')\, dx$$

stationary.
Note that I depends only on ϵ, since x gets integrated out of I, and η and y are fixed functions of x. Since our objective is to make I stationary, we need

$$\frac{dI}{d\epsilon} = 0.$$

However, notice that

$$\left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} = 0$$

since then $\bar{y}(x) = y(x)$, which is stationary as well. We can then use this to differentiate I:

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \int_{x_1}^{x_2} F(x, y, y')\, dx = 0$$

$$\int_{x_1}^{x_2} \frac{\partial}{\partial\epsilon} F(x, y, y')\Big|_{\epsilon=0}\, dx = 0.$$

Since x doesn't depend on ϵ and gets integrated out, we can apply the chain rule of partial differentiation to get

$$\int_{x_1}^{x_2} \left[\frac{\partial F}{\partial \bar{y}}\frac{\partial \bar{y}}{\partial \epsilon} + \frac{\partial F}{\partial \bar{y}'}\frac{\partial \bar{y}'}{\partial \epsilon}\right]\Big|_{\epsilon=0} dx = 0.$$

Since $\bar{y}(x) = y(x) + \epsilon\eta(x)$, it follows that

$$\bar{y}'(x) = y'(x) + \epsilon\eta'(x),$$

and so

$$\frac{\partial \bar{y}}{\partial \epsilon} = \eta$$

and

$$\frac{\partial \bar{y}'}{\partial \epsilon} = \eta'.$$

Plugging this back into our equation we get

$$\int_{x_1}^{x_2} \left[\frac{\partial F}{\partial \bar{y}}\eta + \frac{\partial F}{\partial \bar{y}'}\eta'\right]\Big|_{\epsilon=0} dx = 0.$$

Integrating the second term by parts, we get

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial \bar{y}'}\eta'\ dx = \frac{\partial F}{\partial \bar{y}'}\int_{x_1}^{x_2} \eta'\ dx - \int_{x_1}^{x_2} \left(\int \eta'\right)\frac{d}{dx}\left[\frac{\partial F}{\partial \bar{y}'}\right]\ dx$$

$$= \frac{\partial F}{\partial \bar{y}'}\Big[\eta\Big]_{x_1}^{x_2}\ dx - \int_{x_1}^{x_2} \eta\frac{d}{dx}\left[\frac{\partial F}{\partial \bar{y}'}\right]\ dx.$$

By the boundary conditions we previously established,

$$[\eta]_{x_1}^{x_2} = 0$$

and so this turns into

$$-\int_{x_1}^{x_2} \eta\frac{d}{dx}\left[\frac{\partial F}{\partial \bar{y}'}\right]\ dx.$$

Plugging this back into our original equation we get

$$\int_{x_1}^{x_2} \left[\frac{\partial F}{\partial \bar{y}}\eta + \frac{d}{dx}\left(\frac{\partial F}{\partial \bar{y}'}\right)\eta\right]\Big|_{\epsilon=0} dx = 0$$

$$\int_{x_1}^{x_2} \left[\frac{\partial F}{\partial \bar{y}} + \frac{d}{dx}\left(\frac{\partial F}{\partial \bar{y}'}\right)\right]\eta\Big|_{\epsilon=0} dx = 0$$

At $\epsilon = 0$, $\bar{y} = y$, so this gives

$$\int_{x_1}^{x_2} \Big[\frac{\partial F}{\partial \bar{y}} + \frac{d}{dx}\Big(\frac{\partial F}{\partial y'}\Big)\Big]\eta \, dx = 0.$$

Because η is an arbitrary function, the only way to guarantee this expression holds is if

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\Big(\frac{\partial F}{\partial y'}\Big) = 0$$

, which is the Euler-Lagrange equation.
It is important to note that the Euler-Lagrange equation is a necessary condition for $y(x)$ to make I stationary, but it is not *sufficient*. Thus one cannot assume that, if this equation is true, then $y(x)$ must indeed make I stationary.

5. The Beltrami Identity

Now we highlight the Beltrami Identity, which is a special case of the Euler-Lagrange equation. This occurs when

$$\frac{\partial F}{\partial x} = 0$$

for our functional F across all x. This identity is rarely used in calculus of variations, but it happens to be of use in solving the brachistochrone problem, which will appear later.
Recall that the Euler-Lagrange equation is

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\Big(\frac{\partial F}{\partial y'}\Big) = 0.$$

Multiplying both sides by y' we get

$$y'\frac{\partial F}{\partial y} - y'\frac{d}{dx}\Big(\frac{\partial F}{\partial y'}\Big) = 0. \tag{1}$$

Using the chain rule to take the total derivative of F with respect to x, we get

$$\frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}y' + \frac{\partial F}{\partial y'}y''$$

$$\frac{\partial F}{\partial y}y' = \frac{dF}{dx} - \frac{\partial F}{\partial x} - \frac{\partial F}{\partial y'}y''. \tag{2}$$

Substituting (2) into (1) gives

$$\frac{dF}{dx} - \frac{\partial F}{\partial x} - \Big[\frac{\partial F}{\partial y'}y'' + y'\frac{d}{dx}\Big(\frac{\partial F}{\partial y'}\Big)\Big] = 0.$$

We can simplify the equation in brackets by using the reverse of the product rule, giving us

$$\frac{dF}{dx} - \frac{\partial F}{\partial x} - \frac{d}{dx}\left(y' \frac{\partial F}{\partial y'}\right) = 0$$

$$\frac{dF}{dx} - \frac{d}{dx}\left(y' \frac{\partial F}{\partial y'}\right) = \frac{\partial F}{\partial x}$$

$$\frac{d}{dx}\left(F - y' \frac{\partial F}{\partial y'}\right) = \frac{\partial F}{\partial x}.$$

Using our assumption that

$$\frac{\partial F}{\partial x} = 0,$$

we get

$$\frac{d}{dx}\left(F - y' \frac{\partial F}{\partial y'}\right) = 0.$$

Finally, we integrate both sides with respect to x to get

$$F - y' \frac{\partial F}{\partial y'} = C,$$

where C is a constant of integration.

6. The Brachistochrone Problem

We can use our previous findings to determine the shape of a **brachistochrone curve**, which is the path of quickest descent from one point to another point of lower elevation where the only force acting upon our particle is gravity. This problem was posed by Johann Bernoulli in 1696 and in the end, five mathematicians provided their solutions to the problem, each of whom determined that the curve happens to be a cycloid.

Theorem 1. *(The Brachistochrone Problem) The curve from one point to another that takes the shortest amount of time for a particle to slide along it frictionlessly under the influence of gravity; that is, the brachistochrone curve, is a cycloid.*

Proof. By rearranging the formula for speed, we obtain $t = \frac{d}{s}$; in other words, time equals distance divided by speed. Thus the time taken to travel a given curve from $P1$ to $P2$ can be found using the integral

$$t_0 = \int_{P1}^{P2} \frac{dS}{v}.$$

We can rewrite this using the conservation of energy formula, which states that the sum of the initial potential and kinetic energy equals the sum of the final potential and kinetic energy. We have the following formulas for kinetic and potential energy:

$$KE = \frac{1}{2}mV^2$$

$$PE = mgh.$$

At $P1$, our particle is not moving and thus has no kinetic energy. If $P1$ has a height $h = y$, the particle initially has potential energy mgy. If we specify that $P2$ has a height of $h = 0$ (and thus y can be adjusted accordingly), then our particle at point $P2$ has a potential energy of 0 and a kinetic energy of $\frac{1}{2}mV^2$. Thus we obtain the equation

$$mgy = \frac{1}{2}mV^2$$

$$v = \sqrt{2gy}.$$

Using the Pythagorean Theorem for dS, we can rewrite

$$dS = \sqrt{dx^2 + dy^2}$$

$$= \sqrt{1 + y'^2}dx$$

and so we can plug these values into our original integral to obtain

$$t_0 = \int_{P1}^{P2} \frac{\sqrt{1 + y'^2}}{\sqrt{2gy}}\, dx,$$

meaning our function to be varied is

$$F(y, y') = \sqrt{\frac{1 + y'^2}{2gy}}.$$

Because $F(y, y')$ does not explicitly contain x, $\frac{\partial F}{\partial x} = 0$ and so we can use the Beltrami Identity to vary this function rather than the Euler-Legrange equation.

$$\frac{\partial F}{\partial y'} = y'(y'^2 + 1)^{-\frac{1}{2}}(2gy)^{-\frac{1}{2}}$$

and we can plug this into the Beltrami Identity to get

$$\frac{\sqrt{1+y'^2}}{\sqrt{2gy}} - \frac{y'^2}{\sqrt{1+y'^2}\cdot\sqrt{2gy}} = C$$

$$\frac{1+y'^2}{\sqrt{2gy}\cdot\sqrt{1+y'^2}} - \frac{y'^2}{\sqrt{1+y'^2}\cdot\sqrt{2gy}} = C$$

$$\frac{1}{\sqrt{2gy}\cdot\sqrt{1+y'^2}} = C$$

$$\frac{1}{2gC^2} = y\left(1+\left(\frac{dy}{dx}\right)^2\right).$$

Since the left side of this equation is a constant, we can define

$$k^2 = \frac{1}{2gC^2}$$

to get

$$y\left(1+\left(\frac{dy}{dx}\right)^2\right) = k^2.$$

This turns into the parametric equations

$$x = \frac{1}{2}k^2\left(\theta - \sin\theta\right),$$

$$y = \frac{1}{2}k^2\left(1 - \cos\theta\right).$$

These parametric equations are that of a cycloid and can be easily verified by plugging these parametric equations into our previous result. We can find $\frac{dy}{dx}$ by taking the derivatives of y and x with respect to θ and dividing them:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$= \frac{\frac{1}{2}k^2\sin\theta}{\frac{1}{2}k^2(1-\cos\theta)}$$

$$= \frac{\sin\theta}{1-\cos\theta}.$$

So we plug this into our differential equation to get

$$\Big(1+\Big(\frac{dy}{Dx}\Big)^2\Big)y = y\Big(1+\frac{sin^2\theta}{(1-\cos\theta)^2}\Big)$$

$$= y\Big(\frac{(1-\cos\theta)^2+1-\cos^2\theta}{(1-\cos\theta)^2}\Big)$$

$$= y\Big(\frac{1+\cos^2\theta-2\cos\theta+1-\cos^2\theta}{(1-\cos\theta)^2}\Big)$$

$$= y\Big(\frac{2-2\cos^2\theta}{(1-\cos\theta)^2}\Big)$$

$$= \frac{2y}{1-\cos\theta}$$

$$= \frac{2y}{1-\cos\theta}$$

$$= \frac{1}{2}K^2\cdot 2$$

$$= K^2,$$

which matches the right-hand side of our differential equation. □

This proof may be unsatisfactory to some, however, as it assumes that we already know that the resulting curve is a cycloid and we derive it from there. Clearly, it matches up, but how might one obtain the desired result by going from the differential equation to the parametric equations, and not the other way around? One way to prove the brachistochrone curve problem without this knowledge will be shown here.

Previously we had the equation

$$y\Big(1+\Big(\frac{dy}{dx}\Big)^2\Big) = k^2.$$

.
We will carry on from here.

$$(1 + y'^2)(y) = k^2$$

$$y + y'^2 y = k^2$$

$$y'^2 y = k^2 - y$$

$$y'^2 = \frac{k^2 - y}{y}$$

$$y' = \sqrt{\frac{k^2 - y}{y}}.$$

Recall that $y' = \frac{dy}{dx}$, so

$$\frac{dy}{dx} = \sqrt{\frac{k^2 - y}{y}}$$

$$\frac{dx}{dy} = \sqrt{\frac{y}{k^2 - y}}$$

$$dx = \sqrt{\frac{y}{k^2 - y}} dy.$$

Now we integrate both sides to get

$$x = \int \sqrt{\frac{y}{k^2 - y}}\, dy.$$

This may appear convoluted at first but we can solve this using trig substitution. So let

$$y = k^2 \sin^2\left(\frac{\theta}{2}\right)$$

and therefore

$$dy = k^2 \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right)\, d\theta.$$

So plugging this in we get

$$x = \int \sqrt{\frac{k^2 \sin^2\left(\frac{\theta}{2}\right)}{k^2 - k^2 \sin^2\left(\frac{\theta}{2}\right)}} \left(k^2 \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right)\right)\, d\theta$$

$$x = k^2 \int \sqrt{\frac{\sin^2\left(\frac{\theta}{2}\right)}{1 - \sin^2\left(\frac{\theta}{2}\right)}} \left(\sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right)\right)\, d\theta$$

$$x = k^2 \int \sqrt{\frac{\sin^2\left(\frac{\theta}{2}\right)}{\cos^2\left(\frac{\theta}{2}\right)}} \left(\sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right)\right)\, d\theta$$

$$x = k^2 \int \frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)} \left(\sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right)\right)\, d\theta$$

$$x = k^2 \int \sin^2\left(\frac{\theta}{2}\right)\, d\theta.$$

The half-angle formula for sin tells us that

$$\sin(\frac{\theta}{2}) = \frac{1 - \cos\theta}{2}.$$

Using this, we obtain

$$x = k^2 \int \sin^2\left(\frac{\theta}{2}\right)\, d\theta$$

$$x = k^2 \int \sin^2\left(\frac{1 - \cos\theta}{2}\right)\, d\theta$$

$$x = \frac{k^2}{2} \int \sin^2\left(1 - \cos\theta\right)\, d\theta$$

$$x = \frac{k^2}{2} \int \sin^2\left(1 - \cos\theta\right)\, d\theta$$

$$x = \frac{k^2}{2}\left(\theta - \sin\theta\right) + C$$

For a constant of integration C. This is the x for our parametric equation, and since we set

$$y = k^2 \sin^2\Big(\frac{\theta}{2}\Big),$$

this is our y value. Recall that

$$\sin^2\theta = \frac{1-\cos 2\theta}{2},$$

so since in this case, our input for the sine function is $\frac{\theta}{2}$, we get

$$y = k^2\Big(\frac{1-\cos\theta}{2}\Big).$$

For our x, note that when we previously derived the cycloid, we did not have a constant of integration.
Well, at point A, $x = 0$ and this corresponds to $\theta = 0$ as it is a parametric equation. So:

$$x = \frac{k^2}{2}\Big(\theta - \sin\theta\Big) + C$$

$$0 = \frac{k^2}{2}\Big(0 - \sin 0\Big) + C$$

$$0 = C$$

Thus since $C = 0$, we get our previously desired x. Thus using this method we get the parametric equation we obtained earlier.

7. Other Calculus of Variations Problems

While the brachistochrone problem is perhaps the most famous variational calculus problem, there are plenty others that are far more sophisticated that one might consider exploring if their interest has been piqued thus far. A few of them are mentioned below:

(1) *The Tautochrone Problem:* As explained earlier, this was the problem that introduced the Euler-Lagrange equation. In short, the problem aims to find the curve in which a particle slides down it at a time constant regardless of where on the curve it starts. There are some simple calculus proofs [03], but the actual variational calculus solution is a bit more difficult. As mentioned earlier, the solution to this problem happens to be a cycloid as well. There isn't a correlation between this problem and the brachistochrone problem- it is quite miraculous, actually, that they happen to coincide- but both can be understood intuitively and readers are encouraged to look further into that [10].

It may come to one's attention, however, that cycloids can have an upwards slope at some point (for a visualization of this one can visit the Wikipedia page for the brachistochrone curve [01] which contains a simulation of a cycloid with an upwards slope). For this, we can understand the tautochrone problem through cycloidal pendulums, which are isochronous regardless of their amplitude. A simulation of this can be viewed on the Wikipedia page for the tautochrone curve [08].

(2) *Newton's minimal resistance problem:* This problem marks the beginning of variational calculus. Essentially, the problem tries to find the solid of revolution which experiences the least resistance when traveling in the direction of the axis of revolution. This can be solved using the Euler-Lagrange equation (though Newton used other means to find the answer) and proofs of this can be found in many places online. The most useful solution the author of this paper was able to find is on the Wikipedia page [05] which also happens to use variational calculus (something that many other sites neglect to use since there are other simpler proofs).

(3) *The Euler-Poisson Equation:* The Euler-Poisson equation is just a generalization of the Euler-Lagrange equation. More specifically, the Euler-Lagrange equation considered the function

$$F(x, y, y').$$

However, the Euler-Poisson equation considers functions that take higher derivatives and are in the form of

$$F(x, y, y', ...y^n).$$

The solution is easily derived from the Euler-Lagrange equation and uses almost all the same steps until the end. The author highly encourages interested readers to attempt the derivation on their own but an in-depth, clear explanation can be found here [Pro].

(4) *The principle of least action:* The principle of least action is from Newtonian mechanics, so for the uninformed reader, this may be more difficult. This essentially aims to minimize the action of a system, and once again the Euler-Lagrange equation can be used to determine this. The author lies within the subset of mathematicians who aren't well versed in physics, but for others who may be (or for those who are interested regardless) there are nice explanations on both the Wikipedia page [12] and in section 3.3 of [Kam17].

(5) *Shortest path problem:* As its name implies, the shortest path problem finds the shortest path between two points in $\mathbb{R}^n$. This happens to also be the name of a graph theory problem with the same generic premise but they differ nonetheless. For our purposes, we aim to find $f(x)$ that minimizes the functional

$$I(f) = \int_0^1 \|f\| \, dx$$

where

$$f(0) = x, f(1) = y.$$

From this it is not difficult to guess that this can be solved using the Euler-Lagrange equation. Section 3.3 of [Kam17] contains a nice proof of this, and this is a relatively simpler problem for readers to try earlier on.

Other examples can be found in Section 3.3 of [Kam17] or in [06].

8. Acknowledgements

The author of this paper would like to express his gratitude towards Leonardo Bonnano and Simon Rubinstein-Salzedo for their excellent mentorship and support throughout the process of writing this paper. Additionally, the author thanks Agniv Sarkar, Pico Gilman, and Vedant Janapaty for providing useful suggestions to improve the quality of the paper. Lastly, the author would like to thank the Euler Circle organization for the resources and platform to perform such a project.

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