L^p spaces

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Before going into L^p space...

Here are the thing that will be briefly covered in these slides

- Outer Measure
- Lebesgue Measure
- Lebesgue Integral
- Norm
- L^p space

Sigma Algebra

Denoted as σ-algebra, they are needed in in order for sets to have measures and be well defined.

(i) $X \in \Sigma$ and $\varnothing \in \Sigma$.

(ii) If $B \in \Sigma$, then $X \setminus B \in \Sigma$ i.e. the complement of B in X is also in Σ.

(iii) Any countable intersection or union of elements of Σ is also in Σ.

Lebesgue Outer Measure

It is defined as the following:

Definition

$m^*(A) = \inf \{\sigma(S) \mid S \text{ is a covering of } A \text{ by closed intervals}\}$

Lebesgue Outer Measure

Here are some nice properties:

- $m^*(A) \geq 0$ for any $A \subseteq \mathbb{R}$, and $m^*(\emptyset) = 0$.
- If A and B are disjoint sets, then $m^*(A \cup B) \leq m^*(A) + m^*(B)$.

But there are limitations. For example, one drawback of outer measure is that the union is not always equal to the sum of disjoint sets. We want the outer measure on a sigma algebra to satisfy this property.

Lebesgue Measure

Definition

Carathéodory's criterion

A set $E \subseteq \mathbb{R}^n$ is Lebesgue measurable if for every A in $\mathbb{R}$,
$\lambda^*(A) = \lambda^*(A \cap E) + \lambda^*(A \cap E^c)$

In this case, we define the Lebesgue measure of E, denoted $m(E)$, to be $m(E) = m^*(E)$.

Lebesgue Integral

Suppose f is a bounded function defined on a measurable set E with finite measure. We define the upper and lower Lebesgue integrals, respectively, as

$$U^*(f)_L = \int_E \sup\{\phi(x)\,dx : \phi \text{ is simple and } \phi \geq f\},$$

$$L^*(f)_L = \int_E \inf\{\phi(x)\,dx : \phi \text{ is simple and } \phi \leq f\}.$$

If $U^*(f)_L = L^*(f)_L$, then the function f is called Lebesgue integrable over set E, and the Lebesgue integral of f over set E is denoted by

$$\int_E f(x)\,dx.$$

Norms

A norm is a function that assigns a non-negative real value to vectors or functions. In the context of vector spaces, norms provide a measure of the "length" of a vector. Norm must satisfy certain properties, which are the following:

Properties of Norms

A norm in a vector space V is a function $\|\cdot\| : V \to \mathbb{R}$ with the following properties:

(i) $\|v\| \geq 0$

(ii) $\|v\| = 0$ if and only if $v = \overline{0}$

(iii) $\|cv\| = |c|\|v\|$

(iv) $\|v + w\| \leq \|w\| + \|v\|$

L^p space

Definition

For $p \geq 1$ we define $L^p[a,b]$ to be $L^p[a,b]$ = space of measurable functions f where f is

$$\left(\int_S |f|^p \, d\mu\right)^{1/p} < \infty.$$

The L^p-norm of f, written $||f||_p$, is $\left(\int_a^b |f|^p\right)^{\frac{1}{p}}$. It is not too complicated to prove some properties of norms except the last one regarding triangle inequality.

To verify the final property, we will need to have two inequalities: Holder's Inequality and Minkowski's inequality.

Holder's Inequality

Definition

Let $f \in L^p[a, b]$ and $g \in L^q[a, b]$, where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then $\|fg\|_1 \leq \|f\|_p \|g\|_q$.

Lemma

Let $\alpha, \beta \in (0, 1)$ with $\alpha + \beta = 1$. Then for any nonnegative numbers a and b,

$$ab \leq \alpha a^{\frac{1}{\alpha}} \beta b^{\frac{1}{\beta}}.$$

Minkowski's Inequality

Definition

$\|f+g\|_p \leq \|f\|_p + \|g\|_p$

(i) $(\|f+g\|_p)^{p-1} = (\|(|f+g|^{p-1}\|_q))$

(ii) $|f(x)+g(x)|^p \leq |f(x)| \cdot |f(x)+g(x)|^{p-1} + |g(x)||f(x)+g(x)|^{p-1}$

(iii) $\left\||f+g|\right\|_p \leq \|f\|_p \left\||f+g|\right\|_{p-1} \|g\|_p \left\||f+g|\right\|_{p-1}$