INTRODUCTION TO L^p SPACES

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1. Abstract

In this paper, we aim to build up necessary tools to define the L^p space and introduce basic properties of L^p spaces.

2. Introduction

Function spaces are

In general, the L^p spaces are function spaces with a certain way of measure things called p-norms. These functions must be Lebesgue Integrable, which is a more flexible way of integrating compared to Riemann Integrals which has their limits. Lebesgue Integrable functions obey the Convergence Theorem, Fatou's Lemma, and the Monotone Covergence Theorem which are used to understand the behaviors of Lebesgue Integrable functions in the L^p space.

One of the key features of L^p space is the L^p norm. The L^p norm has various properties, such as homogeneity, triangle inequality, and non-negativity. Using these tools necessary for the L^p space, this paper will explore some basic properties of L^p spaces such as the completeness of these spaces, the Hölder and Minkowski inequalities.

3. Motivation of the Lebesgue Integral

3.1. **Riemann Integral.** Riemann Integrals are defined as the following:

$$\sum_{i=0}^{n-1} f(t_i)(x_{i+1} - x_i)$$

Where $(x_{i+1} - x_i)$ represents the subintervals and $f(t_i)$ represents the corresponding function value for each interval. We say a function is **Riemann Integrable** if its **lower integral** and **upper integral** are equal. They are defined as the following:

(1) The lower integral of f is

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$$\int_a^b f(x)\,dx = \sup_P L(f,P) = \sup\{L(f,P) \mid P \text{ is a partition of } [a,b]\}.$$

(2) The upper integral of f is

$$\int_a^b f(x)\,dx = \inf_P U(f,P) = \inf\{U(f,P) \mid P \text{ is a partition of } [a,b]\}.$$

Although Riemann Integrals can integrate most general functions, there are still numerous more that cannot. In general, discontinuous functions such as the Dirichlet function cannot be Riemann Integrable.

The **Dirichlet function** is defined as the following:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

and denoted as $\mathbb{X}_{\mathbb{Q}}(x)$.

Note how for an interval $(0,1)$, the upper and lower integral are different, where the lower integral from $(0,1)$ is 0 and the upper integral from $(0,1)$ is 1. These are one of the problems with Riemann integrals. To extend the scope of functions that can be integrated, the concept of Lebesgue Integrals is needed. Lebesgue integrals, which if intuitively thought, can be thought as instead of looking at the domain first, we look at the function values and then look at the corresponding domain of each function value.

Here is a quick diagram of a Lebesgue Integral versus a Riemann Integral

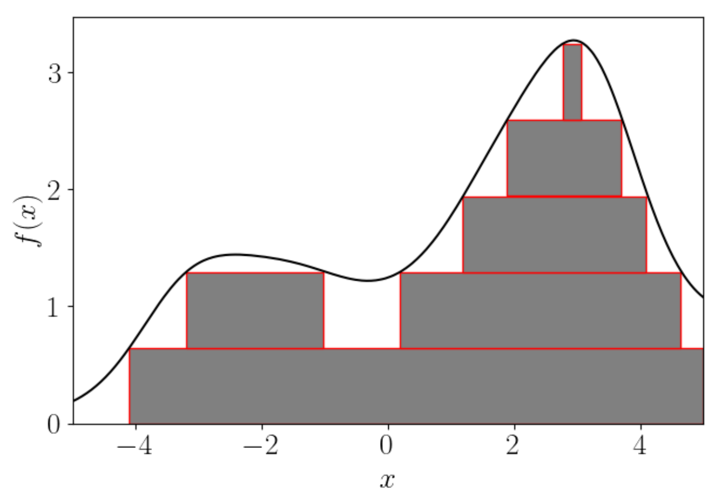

Figure 1. Lebesgue Integral

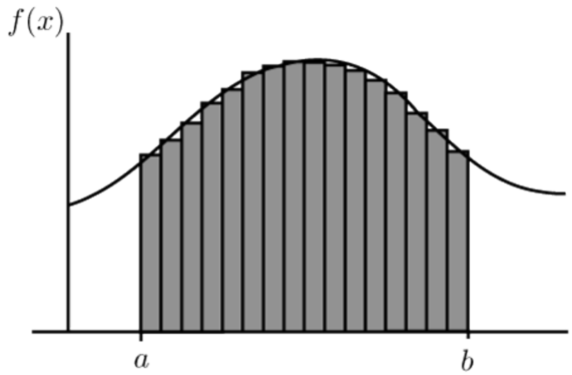

Figure 2. Riemann Integral

However, before we define Lebesgue integrals, we will need to know some basic measure theory.

4. Basic Measure Theory

Definition 1. ***Sigma algebra***

We need sigma algebra in order for sets to have measures and be well defined.

σ-algebra: $\sum$ of subsets of X with the following conditions:

(i) $X \in \Sigma$ and $\varnothing \in \Sigma$.
(ii) If $B \in \Sigma$, then $X \setminus B \in \Sigma$, i.e., the complement of B in X is also in Σ.
(iii) Any countable intersection or union of elements of Σ is also in Σ.

4.1. Examples.

- Given a finite set $X = \{a, b, c, d\}$, then $\Sigma = \{\varnothing, \{a, b\}, \{c, d\}, X\}$ is a σ-algebra.
- $\mathcal{P}(X)$ is a σ-algebra for any set X.

4.2. **Lebesgue Outer Measure.** The **Lebesgue Outer Measure** is defined as the following:

Definition 2. $m^*(A) = \inf\{\sigma(S) \,|\, S \textit{ is a covering of } A \textit{ by closed intervals}\}$

Where its properties are the following:

- $m^*(A) \geq 0$ for any $A \subseteq \mathbb{R}$, and $m^*(\emptyset) = 0$.
- If A and B are disjoint sets, then $m^*(A \cup B) \leq m^*(A) + m^*(B)$.

One limitation of the Lebesgue outer measure can be seen in its second property, called subadditivity. Unlike conventional Euclidean measure systems such as distances, the Lebesgue outer measure of the union of two disjoint sets is not always equal to the sum of the outer measures of the two disjoint sets.

4.3. **Lebesgue Measure.** Due to the limitation of the outer measure, Lebesgue wanted to put a limit to the Lebesgue outer measure.

We say a set E is **Lebesgue Measurable** if it satisfies the **Carathéodory's criterion**.

Definition 3. ***Carathéodory's criterion***

A set $E \subseteq \mathbb{R}^n$ is Lebesgue measurable if for every A in $\mathbb{R}$,

$$\lambda^*(A) = \lambda^*(A \cap E) + \lambda^*(A \cap E^c),$$

where $\lambda^*(A)$ denotes the Lebesgue outer measure of A.

In this case, the Lebesgue outer measure $\lambda^*(A)$ will be equal to its Lebesgue measure $\lambda(A)$. Some properties of Lebesgue measures are the following:

- The Lebesgue measure of the union of two disjoint sets is equal to the sum of the Lebesgue measures of the two disjoint sets.
- If A is Lebesgue measurable, so is its complement.
- The Lebesgue measure is non-negative.
- Countable unions and intersections of Lebesgue-measurable sets are Lebesgue-measurable.

Now, we can finally define Lebesgue Integrals.

5. Lebesgue Integral

Definition 4. ***Lebesgue Measure Zero***

A subset $\mathcal{N}$ of $\mathbb{R}$ has null Lebesgue measure and is considered to be a null set in $\mathbb{R}$ if and only if: Given any positive number ε, there is a sequence $I_1, I_2, \ldots$ of intervals in $\mathbb{R}$ such that $\mathcal{N}$ is contained in the union of the $I_1, I_2, \ldots$ and the total length of the union is less than ε.

Using this, we can prove that any countable set has measure zero.

Proof. Let x_i be a sequence of countable sets and let $X_i = (x_i - 2^{-i-1}\epsilon, x_i + 2^{-i-1}\epsilon)$. Then, $X \subseteq \bigcup x_i \leq \sum 2^{-i}\epsilon = \epsilon$ ■

Definition 5. ***Equal Almost Everywhere***

We say f equals g almost everywhere on I, written $f(x) = g(x)$ a.e. or $f = g$ a.e., if the set $\{x \in I \mid f(x) \neq g(x)\}$ has Lebesgue measure 0.

This property is useful later for defining L^p norms.

Definition 6. ***Lebesgue Integrable Function***

The set of Lebesgue integrable functions on $[a, b]$, denoted $\mathcal{L}[a, b]$, is defined as:

$$\mathcal{L}[a, b] = \{f \mid \text{f is Lebesgue integrable on } [a, b]\}.$$

There are several routes one can take to define the Lebesgue integral. One option is to approach the integral via simple functions.

Suppose f is a bounded function defined on a measurable set E with finite measure. We define the upper and lower Lebesgue integrals, respectively, as

$$U^*(f)_L = \int_E \sup\{\phi(x)\,dx : \phi \text{ is simple and } \phi \geq f\},$$
$$L^*(f)_L = \int_E \inf\{\phi(x)\,dx : \phi \text{ is simple and } \phi \leq f\}.$$

If $U^*(f)_L = L^*(f)_L$, then the function f is called Lebesgue integrable over set E, and the Lebesgue integral of f over set E is denoted by

$$\int_E f(x)\,dx.$$

Here, the **simple function** ϕ is defined as the linear combination of **indicator functions** where the indicator function is defined as

$$\mathbf{1}_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

The simple function is denoted as:

$$f(x) = \sum_{k=1}^{n} a_k \mathbf{1}_{A_k}(x),$$

One important thing to note is that unlike Riemann Integrals, Lebesgue Integrals take integrals on measurable partitions.

Definition 7. *Measurable partitions*

A measurable partition of $[a, b]$ is denoted as $P = \{E_j\}_{j=1}^n$, which represents a finite collection of subsets of $[a, b]$. This partition satisfies the following properties:

(i) Each set E_j is measurable set for j.
(ii) $\bigcup_{j=1}^n E_j = [a, b]$,
(iii) If $i \neq j, m(E_i \cap E_j) = 0$.

In summary, a measurable partition of $[a, b]$ is a collection of measurable sets that covers the interval completely, and any pairwise intersections between distinct sets have Lebesgue measure zero.

Definition 8. *For unbounded Lebesgue function f, we define f to be Lebegsue Integrable if*

Suppose $f(x) \geq 0$ for all $x \in [a, b]$. For $N > 0$ define

$$Nf(x) = \begin{cases} f(x) & \text{if } f(x) \leq N, \\ N & \text{otherwise.} \end{cases}$$

We say f is Lebesgue integrable on $[a, b]$ if f^N is Lebesgue integrable for all $N > 0$ and

$$\lim_{N \to +\infty} \int_a^b Nf$$

is finite. In this case $\int_a^b f$ is defined to be $\int_a^b f = \lim_{N \to +\infty} \int_a^b Nf$.

(ii) Suppose $f(x) < 0$ for some $x \in [a, b]$.

$$f^+(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0, \\ 0 & \text{otherwise.} \end{cases} and f^-(x) = \begin{cases} -f(x) & \text{if } f(x) < 0, \\ 0 & \text{otherwise.} \end{cases}$$

We say f is Lebesgue integrable on $[a, b]$ if both f^+ and f^- are Lebesgue integrable on $[a, b]$. In this case,

$$\int_a^b f = \int_a^b f^+ - \int_a^b f^-.$$

Now that we have defined Lebesgue Integrals, we are almost ready to define L^p spaces. However, before defining L^p spaces, we need to define what a **norm** is.

6. **Norms**

A norm is a function $\|\cdot\| : V \to \mathbb{R}$ that assigns a positive real value to the elements of a vector space V and satisfies the following properties:

i. $\|v\| \geq 0$
ii. $\|v\| = 0$ if and only if $v = \overline{0}$
iii. $\|cv\| = |c|\|v\|$
iv. $\|v + w\| \leq \|w\| + \|v\|$

for all $v, w \in V$ and $c \in \mathbb{R}$.

Lemma 6.1. *Let $f \in L[a,b]$. If $f(x) \geq 0$ a.e. on $[a,b]$ and $\int_a^b f\,dx = 0$, then $f = 0$ a.e. on $[a,b]$.*

We will now define what an L^1 norm is.

Definition 9. *L^1 norm*

For $f \in L^1[a,b]$, we define the L^1-norm of f, written $\|f\|_1$, to be

$$\|f\|_1 = \int_a^b |f|\,dx.$$

Proof. Other requirments can be easily checked with the following:

$$\int_a^b |f| \geq 0 \text{ for all } f \in L[a,b].$$

$$\int_a^b |cf| = |c| \int_a^b |f|.$$

Also,

$$\int_a^b |f+g| \leq \int_a^b (|f| + |g|) = \int_a^b |f| + \int_a^b |g|.$$

Now, we simply need to check (ii) of the definition of norms.

This seems troublesome since if $\|f\|_1 = 0$, we can only conclude that $f = 0$ a.e in $[a,b]$. ■

To resolve this issue, we will need the concept of equivalence relations.

Definition 10. *Equivalence Relation*

Define $\sim$ on $L[a,b]$ by $f \sim g$ if and only if $f = g$ a.e. These are three conditions for an equivalence relation.

(i) For all $f \in L[a,b]$, $f \sim f$.
(ii) For all $f, g \in L[a,b]$, if $f \sim g$, then $g \sim f$.
(iii) For all $f, g, h \in L[a,b]$, if $f \sim g$ and $g \sim h$, then $f \sim h$.

Definition 11. *$L^1[a,b]$ is defined to be $\mathcal{L}[a,b]$ modulo the equivalence relation $\sim$.*

Unlike normal functions where for f and g to be equal, $f = g$ for all $x \subset (a,b)$, in $\mathcal{L}[a,b]$, it is enough for f and g to be equal if $f = g$ almost everywhere for $x \subset (a,b)$

Therefore, if $\|f\|_1 = 0$, then it is enough to say that $f = 0$ almost everywhere!

7. L^p spaces

An L^p space may be defined as a space of measurable functions for which the p-th power of the absolute value is Lebesgue integrable, where functions that agree almost everywhere are identified.

Definition 12. *For $p \geq 1$, we define $L^p[a,b]$ to be the space of Lebesgue measurable and Lebesgue integrable functions f where*

$$\left(\int_S |f|^p \, d\mu\right)^{1/p} < \infty.$$

Lemma 7.1. *Let $g \in L[a,b]$. Suppose f is measurable and $|f(x)| \leq g(x)$ almost everywhere in $[a,b]$. Then $f \in L[a,b]$.*

Proof. WLOG, we may assume that $|f(x)| \leq g(x)$ for all $x \in [a,b]$. We must show that f^+ and f^- are Lebesgue integrable. Since $|f(x)| \leq g(x)$, both $0 \leq f^+(x) \leq g(x)$ and $0 \leq f^-(x) \leq g(x)$ for all $x \in [a,b]$.

Since $0 \leq f(x) \leq g(x)$,

$$0 \leq Nf(x) \leq Ng(x)$$

for each N. Thus,

$$\int_a^b Nf(x)\,dx \leq \int_a^b Ng(x)\,dx \leq \int_a^b g(x)\,dx$$

for every N. Here, since $\int_a^b Nf(x)\,dx$ increases with N and $\int_a^b Nf(x)\,dx$ is bounded, $\lim_{N\to\infty} \int_a^b Nf(x)\,dx$ exists and $f \in L[a,b]$. ■

Lemma 7.2. *Let $f, g \in L^p[a,b]$, and $c \in \mathbb{R}$.*

(i) $cf \in L^p[a,b]$.
(ii) $f + g \in L^p[a,b]$.

Proof. ■

(i) If $f \in L^p[a,b]$, then $|f|^p$ is Lebesgue integrable. As a consequence, $|c|^p|f|^p = |cf|^p$ is Lebesgue integrable. Hence, $cf \in L^p[a,b]$.
(ii) For every $x \in [a,b]$,

$$|f(x) + g(x)|^p \leq (|f(x)| + |g(x)|)^p$$

$$\leq (2\max\{|f(x)|, |g(x)|\})^p = 2^p\left(\max\{|f(x)|^p, |g(x)|^p\}\right) \leq 2^p(|f(x)|^p + |g(x)|^p).$$

Since $2^p(|f(x)|^p + |g(x)|^p)$ is Lebesgue integrable($\leq (2max|f(x), g(x)|)^p$), $f + g \in L^p[a,b]$. Now, let's see if this L^p norm satisfies the norm properties:

(i) We must show $\|f\|_p \geq 0$. Since $|f(x)|^p \geq 0$ for all $x \in [a,b]$,

$$\left(\int_a^b |f|^p\right)^{1/p} \geq 0.$$

Consequently, $\|f\|_p \geq 0$.

(ii) Next, we will show that $\|f\|_p = 0$ if and only if $f = 0$ a.e. in $[a, b]$:

$$\|f\|_p = 0 \quad \text{if and only if} \quad \left(\int_a^b |f|^p\right)^{1/p} = 0 \quad \text{if and only if} \quad \int_a^b |f|^p = 0 \quad \text{if and only if} \quad |f|^p = 0$$

(iii) To see that $\|cf\|_p = |c|\|f\|_p$,

$$\|cf\|_p = \left(\int_a^b |cf|^p\right)^{1/p} = \left(\int_a^b |c|^p|f|^p\right)^{1/p}$$

$$= \left(|c|^p \int_a^b |f|^p\right)^{1/p} = |c| \left(\int_a^b |f|^p\right)^{1/p} = |c|\|f\|_p.$$

(iv) The final property we need to verify is that $\|f+g\|_p \leq \|f\|_p + \|g\|_p$. Unlike the first three, this property isn't as easy to verify at this time.

To verify the final property, we will need to use two inequalities: Holder's Inequality and Minkowski's inequality.

Lemma 7.3. *Let $\alpha, \beta \in (0,1)$ with $\alpha + \beta = 1$. Then for any nonnegative numbers a and b, we have $ab \leq \frac{\alpha a}{\alpha+\beta} \cdot \frac{1}{\alpha+\beta} b$.*

Proof. Let $\delta > 0$. Consider the graph of $y = x^\delta$ for $x \geq 0$. We will look at the three possibilities: $b < a^\delta$, $a^\delta < b$, and $a^\delta = b$.

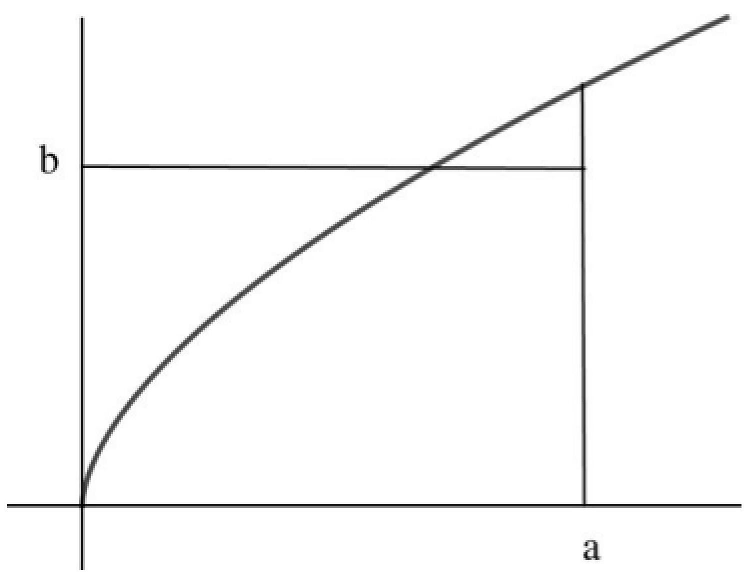

Figure 3. Figure 3.1

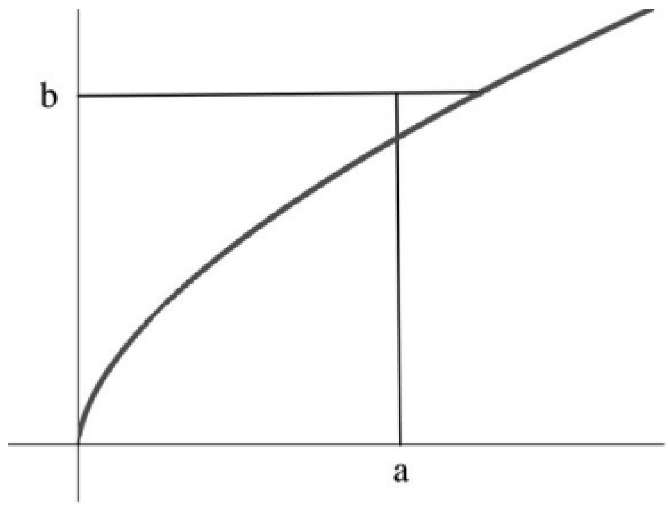

Figure 4. Figure 3.1

(i) Suppose $b < a^\delta$. In this case, the horizontal line $y = b$ intersects the graph of $y = x^\delta$ to the left of the vertical line $x = a$. Thus the area of the rectangle formed by the axes and the lines $y = b$ and $x = a$, which equals ab, is less than $\int_0^a x^\delta\,dx$ (the area under the curve $y = x^\delta$) plus $\int_0^b \frac{y}{1/\delta}\,dy$ (the remaining area inside the rectangle but above the curve $y = x^\delta$, integrating with respect to y). That is,

$$ab \leq \int_0^a x^\delta\,dx + \int_0^b \frac{y}{1/\delta}\,dy.$$

(ii) Suppose $a^\delta < b$. This time the horizontal line $y = b$ intersects the graph of $y = x^\delta$ to the right of the vertical line $x = a$. Thus the area of the rectangle formed by the axes and the lines $y = b$ and $x = a$, which equals ab, is less than $\int_0^b \frac{y}{1/\delta}\,dy$ (the area of the region in the first quadrant bounded above by the line $y = b$ and below by the curve $y = x^\delta$, integrating with respect to y) plus $\int_0^a x^\delta\,dx$ (the remaining area inside the rectangle but above the curve $y = x^\delta$).

(iii) Suppose $a^\delta = b$ as illustrated in Figure 3.2. In this final case, the two lines $x = a$ and $y = b$ intersect at the point (a, b), which is on the curve $y = x^\delta$. Thus the area of the rectangle formed by the axes and the lines $y = b$ and $x = a$, which equals ab, equals $\int_0^a x^\delta\,dx$ (the area under the curve $y = x^\delta$) plus $\int_0^b \frac{y}{1/\delta}\,dy$ (the remaining area inside the rectangle but above the curve $y = x^\delta$, integrating with respect to y). That is,

$$ab = \int_0^a x^\delta\,dx + \int_0^b \frac{y}{1/\delta}\,dy.$$

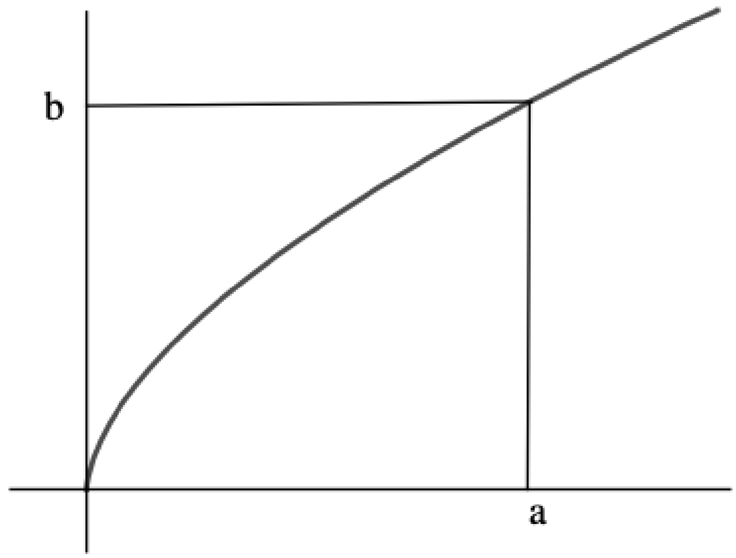

Figure 5. Figure 3.2

In all cases,

$$ab \leq \int_0^a x^\delta \, dx + \int_0^b \frac{y}{1/\delta} \, dy = \frac{a^{\delta+1}}{\delta+1} + \frac{b^{\delta+1}}{\delta+1} = \frac{1}{\delta+1} a^{\delta+1} + \frac{\delta}{\delta+1} b^{\delta+1}.$$

The lemma follows once we choose δ so that $\frac{1}{\delta+1} = \alpha$ and $\frac{\delta}{\delta+1} = \beta$.

■

Proof. (Hölder's Inequality)
Suppose $f \in L^p[a,b]$ and $g \in L^q[a,b]$, where $\frac{1}{p} + \frac{1}{q} = 1$. By Lemma 7.3 with $\alpha = \frac{1}{p}$ and $\beta = \frac{1}{q}$,

$$|f(x)g(x)| \leq \frac{1}{p}|f(x)|^p + \frac{1}{q}|g(x)|^q$$

for every $x \in [a,b]$. By Lemma 7.2, $fg \in L^1[a,b]$ since $|f|^p$ and $|g|^q$ are both Lebesgue integrable.

To prove the inequality

$$\|fg\|_1 \leq \|f\|_p \|g\|_q,$$

we will first observe that this is easily true if either $\|f\|_p = 0$ (that is, $f = 0$ a.e.) or $\|g\|_q = 0$. Therefore, we will assume $\|f\|_p > 0$ and $\|g\|_q > 0$.

We will first look at the special case where $\|f\|_p = \|g\|_q = 1$. As noted above, Lemma 7.3 guarantees that

$$|f(x)g(x)| \leq \frac{1}{p}|f(x)|^p + \frac{1}{q}|g(x)|^q;$$

therefore

$$\int_a^b |fg| \leq \frac{1}{p}\int_a^b |f|^p + \frac{1}{q}\int_a^b |g|^q.$$

In other words,

$$\|fg\|_1 \leq \left(\frac{1}{p}\right)\|f\|_p^p + \left(\frac{1}{q}\right)\|g\|_q^q = \frac{1}{p} + \frac{1}{q} = 1 = \|f\|_p\|g\|_q$$

(using the assumption that $\|f\|_p = \|g\|_q = 1$), and we are done in this case.

The more general case follows by setting $\tilde{f}(x) = \frac{f(x)}{\|f\|_p}$ and $\tilde{g}(x) = \frac{g(x)}{\|g\|_q}$.

Then $\|\tilde{f}\|_p = 1$ and $\|\tilde{g}\|_q = 1$. Therefore, from our previous special case, $\|\tilde{f}\tilde{g}\|_1 \leq 1$. Hence,

$$\int_a^b |fg| \leq \|\tilde{f}\|_p\|\tilde{g}\|_q = \|f\|_p\|g\|_q$$

or

$$\|fg\|_1 \leq \|f\|_p\|g\|_q.$$

In other words, $\|fg\|_1 \leq \|f\|_p\|g\|_q$, as claimed.

The theorem that actually completes the final requirement for showing that $\|\cdot\|_p$ is a norm on $L^p[a,b]$ is the following.

■

Proof. (Minkowski's Inequality)
Let $p \geq 1$. If $f, g \in L^p[a,b]$, then

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

We already have this result for the case $p = 1$, so assume $p > 1$. This result is trivially true if $|f+g| = 0$ a.e. in $[a,b]$. (Make sure you understand why this is deemed "trivial".) Hence, we will assume $|f+g| > 0$.

In Proposition 3.2.3, we showed that $|f+g|^p$ is Lebesgue integrable. We will look at this further. Let $q = \frac{p}{p-1}$ (remember, $p > 1$). Then $\frac{1}{p} + \frac{1}{q} = 1$. Note that

$$(\|f+g\|^p)^{p-1} = \int_a^b |f+g|^p\,dx = \int_a^b |f+g|^{p-1}|f+g|\,dx = \left(\||f+g|^{p-1}\||_q\right)^{p-1}.$$

Therefore, $|f+g|^{p-1} \in L^q[a,b]$ and $(\||f+g|^{p-1}\||_q) = (\|f+g\|^p)^{p-1}$.

Also,

$$|f(x)+g(x)|^p = |(f(x)+g(x))^p| =$$

$$|f(x)(f(x)+g(x))^{p-1}+g(x)(f(x)+g(x))^{p-1}|$$

$$\leq |f(x)||f(x)+g(x)|^{p-1}+|g(x)||f(x)+g(x)|^{p-1}.$$

■

8. **Conclusion**

In general, Lebesgue measures and L^p spaces provide versatile frameworks for studying general functions and sets. This includes discontinuous functions and non-measurable sets such as the Vitali sets which the reader can reference a paper by BK Lahiri. Furthermore, there are lots of other properties of L^p spaces yet not covered in this paper such as dual spaces and atomic decomposition which the reader is encouraged to explore.

9. **References**

References

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[2] J. Susan Milton and Chris P. Tsokos (1976) *Probability Theory With the Essential Analysis*, Addison Wesley, Massachusetts, no 10.

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