Pattern Avoidance

Arjun Bhobe

Euler Circle

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Introduction to a Pattern Avoidance

Definition

A permutation $p \in [n]$ is said to *follow* a permutation pattern $q \in [m]$ if there exists a subsequence of p such that the subsequence matches the relative ordering or "pattern" of q. Otherwise we say p *avoids* q.

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1653724 follows the 3-permutation 132 because we can select the subsequence 374 which follows the same pattern as 132.

Example

The permutation 1375642 avoids 213 because we cannot select a subsequence that follows the ordering of 213.

Visual Examples

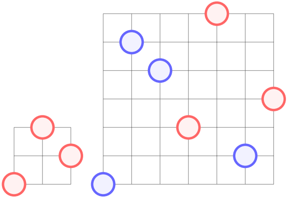

Figure. The pattern 132 and the permutation 1653724.

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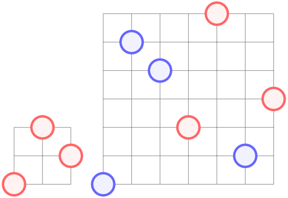

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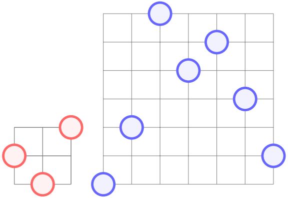

Figure. The pattern 213 and permutation 1375642.

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Definition

The complement of permutation $p \in [n]$ is p^c where $p_i^c = n + 1 - p_i$.

List of 3-patterns

There are a total of 3! or 6 3-patterns. Which are:

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It would be nice if we could reduce the amount of permutations we have to count. Which we can do with complements and reverses. What's interesting is that it turns out that all 3-patterns are equivalent.

Equivalent Patterns

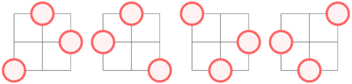

Figure. 132, its reverse, complement, and reverse of its complement.

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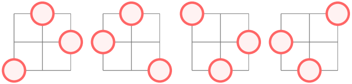

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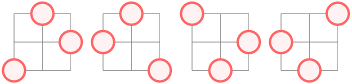

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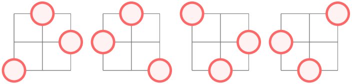

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Proof.

For all $p \in S_n(q)$, $f(p) = p'$. If q' is q^r or q^c, then p' is p^r or p^c, respectively. We know p' avoids q' because both are equivalent rotations of p and q which have an avoidance. ∎

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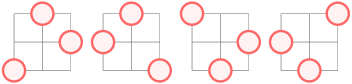

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From complements and reverses, we now know

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$$S_n(132) = S_n(231) = S_n(312) = S_n(213),$$

and

$$S_n(123) = S_n(321).$$

We've unified the 6 permutations into two different groups. However, it turns out we can unify *all* 3-patterns.

Unifying 3-patterns pt. 2

Theorem

$|S_n(123)| = |S_n(132)|$

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Given permutation p, p_i is a *left-to-right minima* if and only if for all $j < i$, $p_j > p_i$.

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Definition

Given permutation p, p_i is a *left-to-right minima* if and only if for all $j < i$, $p_j > p_i$.

Proof.

Given a 123 avoiding permutation p, we will construct a 132 avoiding permutation p'. First, fix all the left-right minima in their positions and delete the remaining entries. Moving left-to-right, in every open position, add the least available entry greater than the closest left-to-right minima on the left.

Unifying 3-patterns pt. 3

We begin with the 123 avoiding permutation 68371542. Deleting all the non-minima, we get 6-3-1—. Adding entries appropriately to the rule, we get 67341258.

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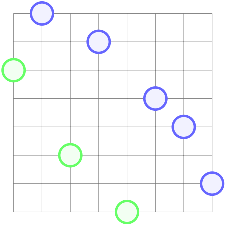

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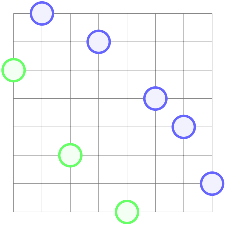

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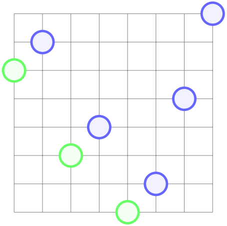

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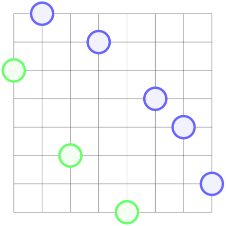

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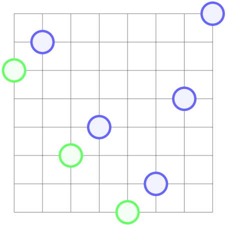

Figure. 67341258

Proof.

We complete the bijection with an inverse function. Given a 132-avoiding permutation, we remove all the non-minima and return elements in decreasing order, avoiding 123. ∎

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Now that we've shown equivalence between all 3-patterns.
Let's actually enumerate the pattern avoidance.

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Lemma

The recurrence

$$C_n = \sum_{i=1}^{n-1} C_{i-1} C_{n-i}$$

is satisfied if and only if C_n is the nth Catalan number.

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Theorem

$$|S_n(132)| = C_n$$

Enumerating 3-patterns cont.

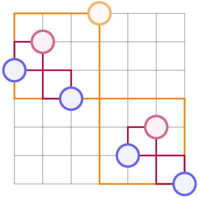

Figure. 5648231, $n = 8$, $i = 4$

Proof.

Let $p \in [n] \cap S_n(132)$. Let n be at the ith position. All elements preceding n should be greater than all elements subsequent of n. There are $i - 1$ elements preceding n and $n - i$ elements succeeding n. We need to repeat the arrangement on a smaller scale, being $C_{i-1}C_{n-i}$. Loop through all possibilities of i and get the Catalan recurrence.

4-patterns

There are 24 4-patterns. Which we can reduce down to the following 8:

$$1234, 1243, 1324, 1342, 1423, 1432, 2143, 2413.$$

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Let's look at some of their direct results.

n	1	2	3	4	5	6	7	8
$\lvert S_n(1342)\rvert$	1	2	6	23	103	512	2740	15485
$\lvert S_n(1234)\rvert$	1	2	6	23	103	513	2761	15767
$\lvert S_n(1324)\rvert$	1	2	6	24	103	513	2762	15793

4-patterns cont.

Question

What makes patterns easier or harder to avoid?

4-patterns cont.

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Question

If for some n, $S_n(q_1) < S_n(q_2)$*, is* $S_N(q_1) < S_N(q_2)$ *for all* $N > n$*?*

Stanley-Wilf Conjecture

Theorem

For any permutation q, there exists a constant c_q, such that for all n,

$$S_n(q) \leq c_q^n.$$