PATTERN AVOIDANCE

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Abstract. Pattern avoidance is the counting of amount n-permutations that avoid a given q-permutation or pattern. This paper will try to include more visuals to build intuition. This kind of visual was briefly introduced by Bóna in Chapter 4 of his book, *Combinatorics of Permutations* [B1́6], but largely overlooked for most of the material.

1. Introduction

We will begin by introducing the concept of a pattern and pattern avoidance. Consider a permutation p of the first n natural numbers and a permutation q of the first m natural numbers. We say n *follows* the pattern q if there exists subsequence of n such that it follows the same ordering or "pattern" as q. If no such subsequence exists, then we say n *avoids* q.

Example. Consider the 7-permutation 1653724 and the 3-permutation 132. Now based on the conventions above, notice that 1653724 follows 132, because we can select the subsequence 374 which follows the same pattern as 132 since the first digit is the least in the subsequence, the second the largest, and the third the middle. Note that this specific sequence contains numerous other subsequences that follow this pattern. Looking at Figure 1, we can visually compare the permutation and the pattern it follows.

As a brief explainer, the x-value of a point represents the position in the sequence while the y-axis represents the value of the sequence at that position. We can visually see that in the red nodes highlighted in the second grid follow the original pattern in the first grid.

Example. Similarly, we can say the permutation 1375642 avoids 213. Observe that in the graph in Figure 2 we cannot select any three points that matches the shape of the given pattern.

Now we will introduce a formal definition of what a pattern avoidance is.

Definition 1.1. Let $q = (q_1, q_2, \dots, q_k) \in S_k$ and $p = (p_1, p_2, \dots, p_n) \in S_n$. We say p follows q if there exists a sequence $(p_{i_1}, p_{i_2}, \dots, p_{i_k})$, such that $i_1 < i_2 < \cdots < i_k$ and $p_{i_a} < p_{i_b}$ if and only if $q_a < q_b$. If no such sequence exists, p avoids q.

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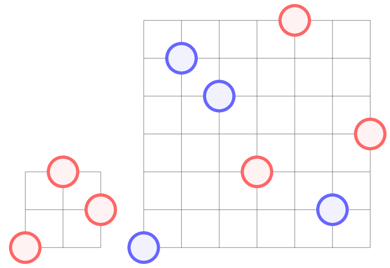

Figure 1. The pattern 132 and the permutation 1653724

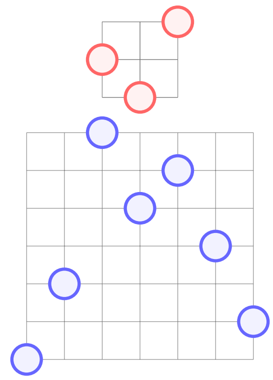

Figure 2. The pattern 213 and permutation 1375642.

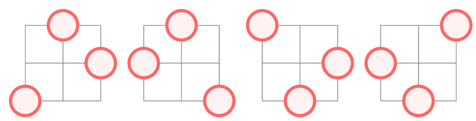

Figure 3. The permutation, its reverse, its complement, and the reverse of its complement.

It turns out that counting patterns that are followed is a tedious task. Instead the focus will be more on counting permutations avoid a specific pattern.

Definition 1.2. Define $S_n(\mathcal{M})$ to be the set of permutations in S_n that avoid all patterns in the set $\mathcal{M}$.

We will primarily be focusing on *singleton* classes of permutations where $|\mathcal{M}| = 1$ or there is just one pattern. In these scenarios we will typically use the single permutation q to represent $\mathcal{M}$.

Definition 1.3. If $|S_n(\mathcal{M})| = |S_n(\mathcal{M}')|$, for all n, then we say $\mathcal{M}$ and $\mathcal{M}'$ are *Wilf-equivalent* or $\mathcal{M} \overset{W}{\sim} \mathcal{M}'$.

2. 3-patterns

The first non-trivial insights on pattern avoidance come from patterns of size three. There area total of 3! or 6 different 3-patterns:

$$123, 132, 213, 231, 312, 321.$$

We will try to establish equivalences between as much patterns as possible.

Definition 2.1. For a permutation $p = (p_1, p_2, \ldots, p_n)$. We define the *reverse* as $p^r = (p_n, p_{n-1}, \cdots, p_2, p_1)$

Definition 2.2. Similarly we define the *complement* of p as $p^c = (n+1-p_1, n+1-p_2, \ldots, n+1-p_n)$

Consider $q = 132$, then $q^r = 231$, $q^c = 312$ and $(q^r)^c = (q^c)^r = 213$. We graph this specific case in Figure 3 Based on this we can prove a significant result about 3-patterns.

Theorem 2.3. $S_n(132) = S_n(231) = S_n(312) = S_n(213)$

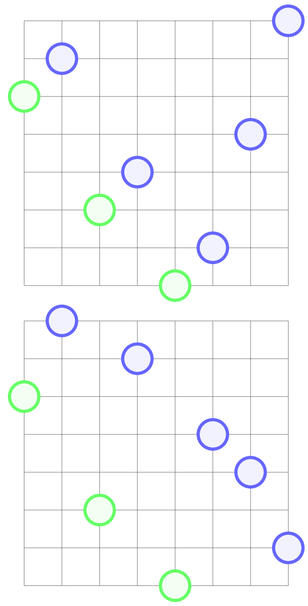

Figure 4. 67341258 and $f(67341258)$

Proof. Observe that q^r is an x-axis reflection of q. Similarly, q^c is a y-axis reflection and $(q^r)^c$ is both an x and y axis reflection. Therefore, we can conclude if and only if permutation p avoids q, then p^r avoids q^r, p^c avoids q^c, and finally $(p^r)^c$ avoids $(q^r)^c$. Now we can create a bijection between every p that avoids q to every p that avoids q^r, q^c, and $(q^c)^r$. ∎

Among the 6 3-patterns, we have established symmetry between 4 of them, leaving 123 and 321. Right off the bat, we can see that $S_n(123) = S_n(321)$, since they are both complements and reverses of each other. We can unify all 6 patterns by creating a bijection between 123 and 132.

Definition 2.4. An element p_i is described as a *left-to-right minima* of the permutation p if for all $k < i$, $p_i < p_k$.

Theorem 2.5. $S_n(123) = S_n(132)$

Proof. Given a 132 avoiding permutation p, we will prove we can construct a unique 123 avoiding permutation using the function $f(p)$. First we "fix" all the left-to-right minima or essentially deleting all elements in the sequence except the minima and retaining their positions. Next we add back the remaining numbers in decreasing order, which certainly avoids the increasing 123 pattern.

We show f is bijective by showing it has an inverse g which will return a unique 132-avoiding permutation. First. we fix the left-to-right minima and then add the least available element greater than closest left-to-right minima on the left. Notice that this makes it impossible to create a 132 pattern because for each entry before the next minima, the adjacent entry is the lowest entry greater than previous entry preventing a possible middle value as the 3rd element in the pattern from being selected since it must be the 2nd element. This can be visually seen in Figure 4.

∎

Now that we have shown a sort of equivalence between all 3-patterns. It will be worth generalizing enumerations. Note that this task of enumeration becomes more tedious as we

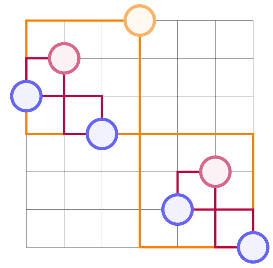

Figure 5. A somewhat symmetric example illustrating the Catalan recurrence of 132-avoiding permutations.

increase the size of q, but the 3-pattern yields a particularly beautiful result in relation to Catalan numbers. This is best done with the pattern 132.

Theorem 2.6. *$S_n(132) = C_n$ where C_n denotes the nth Catalan number.*

Lemma 2.7.

$$C_n = \sum_{i=1}^{n-1} C_{i-1}C_{n-i}$$

It follows the if we can show that $S_n(132)$ satisfies this recurrence, $S_n(132)$ is the nth Catalan number.

Proof. Let us consider n (the maximal element) is p_i in a 132 avoiding permutation p. All elements preceding n must greater than all elements after n. Otherwise, we introduce a contradiction by creating 132 permutation. This means the set of elements preceding n is $\{n-1, n-2, \ldots, n-i+1\}$ and set of elements succeeding n is just $[n-i]$. In order to continue avoiding 132, these subsequences will need to go through the same process. Which means going through all possibilities given the predefined i, $C_{i-1}C_{n-i}$ is the number of 132-avoiding permutations. Now we must add up all possibilities of i, effectively representing the Catalan recurrence, and completing the proof. ■

3. 4-patterns

Now that we've enumerated 3-patterns, naturally we'd look more at 4-patterns. This is where we begin to see the more interesting results of pattern avoidance (i.e. symmetries start to break). Right of the bat, we see there are a total of 24 4-patterns. Through inverses, complements, and both at the same time we can reduce the total number of patterns down to the following 8:

$$1234, 1243, 1324, 1342, 1423, 1432, 2143, 2413.$$

We can still reduce the amount of cases to consider. To do that, we will use an alternative means of referring to a permutation using matrices.

Definition 3.1. A permutation p of length n can be written as the following *permutation matrix*

$$P = \begin{bmatrix} P_{11} & P_{12} & \ldots & P_{1n} \\ P_{21} & P_{22} & \ldots & P_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ P_{n1} & P_{n2} & \ldots & P_{nn} \end{bmatrix}.$$

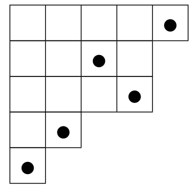

Figure 6. The (5,4,4,2,1) Young diagram with a transversal.

Where $P_{ij} = 1$ if and only if $p_i = j$. Otherwise, $P_{ij} = 0$.

Definition 3.2. The *inverse* p' of a permutation p is defined as the permutation equivalent to the transpose of the permutation matrix of p. Since it is a square matrix, $P'_{ij} = P_{ij}$.

Furthermore, we can extend our definition of pattern avoidance to this method as well.

Definition 3.3. A permutation p in the form of a permutation matrix P is said to follow or contain the pattern q of size m if there exists two subsets of matrix indices $[n]$, $R = \{r_1 < r_2 < \cdots < r_m\}$ and $C = \{c_1 < c_2 < \cdots < c_m\}$ such that

$$\begin{bmatrix} P_{r_1,c_1} & P_{r_1,c_2} & \dots & P_{r_1,c_m} \\ P_{r_2,c_1} & P_{r_2,c_2} & \dots & P_{r_2,c_m} \\ \vdots & \vdots & \ddots & \vdots \\ P_{r_m,c_1} & P_{r_m,c_2} & \dots & P_{r_m,c_m} \end{bmatrix} = C$$

Notice that a permutation matrix is the same thing as if our diagrams were rotated 90 degrees clockwise we replaced every point with 1 if it had a circle and a 0 otherwise.

Now the inverse of the permutation equates to a reflection over the line $y = x$.

Proposition 3.4. $S_n(q) = S_n(q')$

Proof. We can create a bijection between all permutations p that avoid q and all permutations p' that avoid q'. We know that q' is the inverse of q, so we define our bijective function f as $f(p) = p'$ where for all $p \in S_n(q)$, p' is the inverse of p. This p' avoids q' because both are reflections of p and q respectively and p avoids q, maintaining the same pattern avoiding shape. We define the inverse g as $g(p') = (p')' = p$ for all $p' \in S_n(q')$. A transpose/inverse is symmetric so its an inherent property of the function to be bijective. ∎

Based of this we get that $S_n(1423) = S_n(1342)$, the former we shall remove from our list.

Next we will use a special case of the theorem by Backelin, West, and Xin [BWX07]. This is arguably one of the most significant results in the literature of higher-order pattern avoidance as it generalizes significant subset of permutation patterns.

Theorem 3.5. *Let k be any positive integer and q be a permutation of the set $\{k+1, k+2, k+3, \dots, k+r\}$. Then for all positive integers n,*

$$S_n(123\dots kq) = S_n(k\dots 321q).$$

The proof for this requires some new background we have not defined before, specifically concerning Young diagrams. A Young diagram λ is a grid board of a total of n Each row λ_i of this diagram has at least as many boxes as λ_j if and only if $i < j$.

Definition 3.6. A *transversal* L of a Young diagram is an assignment of 1s and 0s to each square on the board such that each row and column of the board contains exactly one 1.

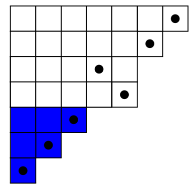

Figure 7. Coloring the Young diagramms as by 3.12

We will now extend our definition of pattern avoidance to Young diagrams.

Definition 3.7. The Young diagram λ is said to contain the $k \times k$ permutation matrix M if there are k rows $r_1 < r_2 < \cdots < r_k$ and k columns $c_1 < c_2 < \cdots < c_k$ and k rows $r_1 < r_2 < \cdots < r_k$ such that $(r_i, c_j) \in \lambda$ and $(r_i, c_j) \in T$ if and only if $M_{i,j} = 1$. Otherwise we say λ avoids M

Note that for a transversal to even be able to exist, $\lambda_i \geq n - i + 1$.

Definition 3.8. Let $S_\lambda(M)$ to be the number of transversals of λ that avoid M.

Definition 3.9. We say permutations M and M' are *shape-Wilf-equivalent* if for all Young diagrams λ, if $|S_\lambda(M)| = |S_\lambda(M')|$. This is represented as $M \overset{sW}{\sim} M'$.

We can see how shape-Wilf-equivalence implies standard Wilf-Equivalence. Now we will restate our main theorem with our new definitions.

Theorem 3.10. *Let M and M' be the following permutation matrices:*

$$M = \begin{bmatrix} I_t & 0 \\ 0 & A \end{bmatrix} M' = \begin{bmatrix} J_t & 0 \\ 0 & A \end{bmatrix}.$$

Where $I_t = 12\ldots t$ and $J_t = t\ldots 21$. Then $M \overset{sW}{\sim} M'$.

The theorem can be derived from the following general propositions.

Proposition 3.11. *For all $t > 0$, $I_t \overset{sW}{\sim} J_t$.*

Proposition 3.12. *Let C and D be two square matrices of order n and let M and M' be the following*

$$M = \begin{bmatrix} C & 0 \\ 0 & A \end{bmatrix}, M' = \begin{bmatrix} D & 0 \\ 0 & A \end{bmatrix}.$$

If $C \overset{sW}{\sim} D$, then $M =\overset{sW}{\sim} M'$.

Proof. Since we assume $C \overset{sW}{\sim} D$, let Π_π be a bijection from $S_\pi(C)$ to $S_\pi(D)$, for all Young diagrams π. For this proof, we will construct a bijection α between $S_\lambda(M)$ and $S_\lambda(M')$. Assume $N \in S_\lambda(M)$.

(1) For any square $(i, j) \in \lambda$, if the subboard below and right of it contains A, then color it white, else color it blue.
(2) Color the entire row and column blue for all the squares that contain 1 and are colored blue. Delete all the blue squares.
(3) Let the remaining white board be denoted as π and its transversal L. We transform L to $\Pi_\pi(L)$., avoiding D. This implies the whole board altogether $\alpha(N)$ avoids M'.

Before resolving the caveats we should note that what this is doing is saying that if there is a subboard that contains A, we can isolate it and transform the rest of the board which avoids C to a board that avoids D, implying that the whole transformed board avoids M'. Now we'll show that the isolated board itself is a Young diagram allowing it to undergo the entire transformation. In step 1, assume a given square (r, c) is colored white, then all squares to the top and left of it also should be colored white as well since they also contain a subbboard that contains A. This creates the white rectangle board with partial transversal from $(1, 1)$ to (r, c), a valid Young diagram but not a valid transversal. However, in step 2, we color the respective rows and columns with blue 1s blue as well, leaving all the white board to have exactly one 1 in each row and column, creating the valid transversal.

To complete the bijection we will show the inverse $\alpha^{-1}(N')$. Note that since Π_π is a bijection, we can use its inverse Π_π^{-1}. Assume $N' \in S_\lambda(M')$.

(1) For any square $(i, j) \in \lambda$, if the subboard below and right of it contains A, then color it white, else color it blue.
(2) Color the entire row and column blue for all the squares that contain 1 and are colored blue. Delete all the blue squares.
(3) Let the remaining white board be denoted as π and its transversal L'. We transform L' to $\Pi_\pi^{-1}(L')$, avoiding C. This implies the whole board altogether $\alpha(N')$ avoids M.

The procedure is the same up to the final step for both α and α^{-1}. However, the only difference in step 3 is that Π_π and Π_π^{-1} change the positions of the 1s on the board but maintaining the same coloring implying the boards are equivalent.

Backelin et al. does note that we can extend the proposition of A beyond the singleton case with a class of matrices without any additional proof. However, they also note that a major caveat of this proposition is that it exclusively applies to shape-Wilf-equivalence and breaks with standard Wilf-equivalence. The example they provide is that $1234 \overset{W}{\sim} 2143$, but $123456 \overset{W}{\not\sim} 214356$ since $|S_9(123456)| = 344,837$ which is not equal to $S_9(214356) = 344,838$. ■

Moving on to Proposition 3.11, Backelin et al. instead proves an equivalent proposition:

Proposition 3.13. *Let F_t be the following:*

$$F_t = \begin{bmatrix} J_{t-1} & 0 \\ 0 & 1 \end{bmatrix}.$$

Then for all $t > 0$, $F_t \overset{sW}{\sim} J_t$.

Proof of equivalence of Propositions 3.11 and 3.13. If we assume Proposition 3.11 to be true, then $J_t \overset{sW}{\sim} I_t$. We also know that $I_{t-1} \overset{sW}{\sim} J_{t-1}$. Now using 2.3 where $C = I_{t-1}, D = J_{t-1}, A = I_1.$, we show $I_t \overset{sW}{\sim} F_t$.

Now let's show the other direction, assume 3.12 is true. Then we define for all $0 \leq k \leq t$, $G_{t,k}$ as

$$G_{t,k} = \begin{bmatrix} J_{t-k} & 0 \\ 0 & I_k \end{bmatrix}.$$

Now for all $0 \leq k \leq t$, we can use proposition 2.3 where $C = J_{t-k}$, $D = F_{t-k}$, and $A = I_k$ to show that $G_{t,l} \overset{sW}{\sim} G_{t,k+1}$ eventually showing that $G_{t,0} \overset{sW}{\sim} G_{t,t}$, which is proposition 3.11.

■

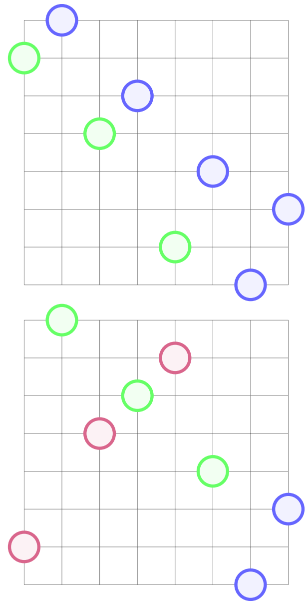

Figure 8. Selecting the 321-pattern from the 123-avoiding permutation over 2 iterations.

Now that we have determined the equivalence between the two propositions, Backelin et al. states that the proof for Proposition 3.13 is a bijection from $S_\gamma(F_t)$ and $S_\gamma(J_t)$.

They proceed to give the following algorithm. Let G the $t \times t$ submatrix that is isomorphic to (or follows) J_t, and let $\theta(G)$ be the submatrix that follows F_t but in the same squares as G.

We will now use the following algorithm as our bijection from $S_\gamma(F_t)$ with transversal L to $S_\gamma(J_t)$.

(1) If L contains no J_t, end.
(2) Find the highest square a_1 containing a 1, such that there is a J_t in L where a_1 is the leftmost 1.
(3) Find the leftmost square a_2 counting a 1, such J_t in L where a_1 and a_2 are the leftmost 1s.
(4) Repeat step 3 one-by-one for a_3 through a_t, to get our set G.
(5) Leaving all other squares fixed, replace G with $\theta(G)$
(6) Repeat the procedure until no J_t is left, leaving us with $T \in S_\gamma(J_t)S$.

If we did undergo steps 2-5, (i.e L contains J_t), then we represent the transformation as $\phi(L)$. In Figure 8 we can see how we select the highest 1 element in the permutation matrix (leftmost in graph), and repeatedly select the leftmost element in the matrix (lowest in the graph).

Now we'll construct the inverse bijection from $S_\gamma(J_t)$ with transversal T to $S_\gamma(F_t)$.

(1) If T contains no F_t, end.
(2) Find the lowest square b_t containing a 1, such that there is a F_t in T where b_t is the rightmost 1.
(3) Find the rightmost square $bt - 1$ containing a 1, such that there is an F_t in T where b_t and b_{t-1} are the leftmost 1s.
(4) Repeat step 3 one-by-one for b_{t-2} through b_1, to get our set H isomorphic to F_t.

(5) Leaving all other squares fixed, replace H with $\theta^{-1}(H)$.
(6) Repeat the procedure until no F_t is left, leaving us with $L \in S_\gamma(F_t)S$.

4. Acknowledgements

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