Balanced incomplete block designs

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Euler Circle

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4		2	
	2	4	3
2	1	3	
	4		2

Figure: 4x4 Latin square

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• *Balanced incomplete block design:* a regular, uniform, balanced design that is not *complete*, i.e. $k < v$.

Examples of block designs

Only regular, uniform and balanced designs are called (v, k, λ) designs, since given that they are regular, uniform and balanced, b and r can be derived directly from the values of v, k and λ.

$$Formula : b = \frac{vr}{k} = \frac{\lambda v(v-1)}{k-1}$$

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For example, a BIBD with parameters $(20, 16, 5, 4, 1)$ is

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Figure: BIBD with parameters (20,16,5,4,1)

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<u>Proof of the second condition:</u>

- consider a certain point x in the design, which appears in λ blocks $\implies$ total number of appearances of x is $r(k = 1)$
- $\forall\ y \in V$ such that $y \neq x$, the pair (x, y) appears λ different blocks $\implies$ number of appearances of the pair (x, y) is $\lambda(v-1)$
- LHS and RHS are equal since they are both ways to count the appearances of point x

Steiner & Kirkman triple systems

Triple systems

A regular balanced design in which every block has cardinality 3, i.e. $k = 3$. A Steiner triple system is a triple system in which $\lambda = 1$, i.e. every pair of points appears in exactly one block.

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In a boarding school, there are fifteen schoolgirls who always take their daily walk in rows of threes. How can it be arranged so that each schoolgirl walks in the same row with every other schoolgirl exactly once a week?

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If every schoolgirl walks in the same row with every other schoolgirl exactly once a week, then every possible pair of points must appear together in a block exactly once, which means $\lambda = 1$.

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• A BIBD is resolvable if its blocks can be partitioned into sets, each of which is a partition of the point set. A resolvable STS is called a Kirkman triple system.

An STS is usually constructed using Latin squares.

• STS(15) $\implies$ 2

$V = \{u_1, u_2, u_3, u_4, u_5, v_1, v_2, v_3, v_4, v_5, w_1, w_2, w_3, w_4, w_5\}$

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Figure: Arrangements of 15 schoolgirls satisfying the problem conditions

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Note that this is a STS and not a KTS, since it cannot be partitioned into groups of five girls each such that every girls appears exactly once in each group.

t-designs

Definition

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Figure: A 3-(10,4,1) design, $t=3$

Important Results concerning BIBDs

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Keevash's Theorem

Given k and t, there is an integer n such that for every $n < v$ that satisfies the following conditions:

- $v \in \mathbb{Z}$
- $all 0 \leq i \leq t-1$, $\binom{k-i}{t-i}$ is a divisor of $\lambda\binom{v-i}{t-i}$

a $t-(v, k, 1)$ design exists.

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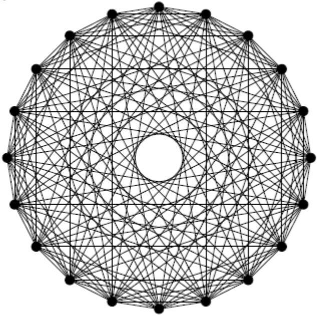

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- elimination of variability in experimental conditions
- identifying the correlation between dependent and independent variables
- improves accuracy of statistical analysis

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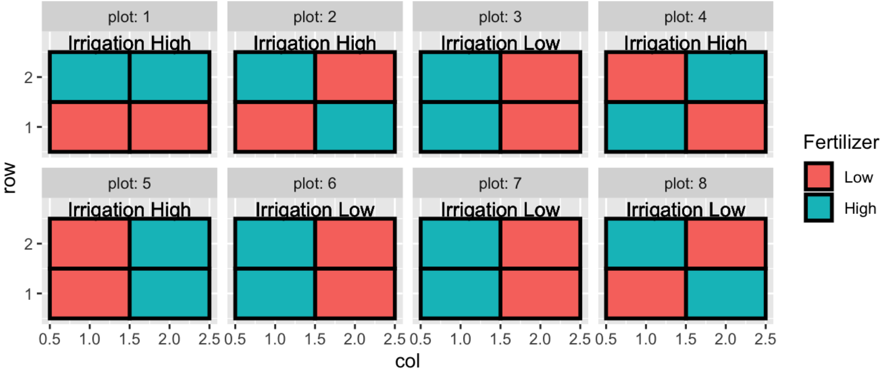

Figure: Agricultural field with varying irrigation and randomised blocking of fertilizer supply

Thank you!