AN INTRODUCTION TO BALANCED INCOMPLETE BLOCK DESIGNS

ANUSHKA TONAPI

Abstract. In this expository paper, our goal is to present a comprehensive introduction to balanced incomplete block designs. We elaborate on different types of designs such as symmetric BIBDs (SBIBDs), and discuss the construction of designs. We also state and prove the conditions of existence of a balanced incomplete block design. Subsequently, relations between the parameters of a BIBD are explored and proved, and we state the Bruck-Ryser-Chowla Theorem. Next, we consider alternate ways to represent BIBDs through incidence matrices and graph colouring. We also discuss Steiner-Kirkman triple systems, t-designs, and the Ray-Chaudhuri and Wilson theorem. Finally, we discuss the application of BIBDs to statistical analysis and experimental design.

Contents

1. Introduction

Combinatorial design is a field in combinatorics which is concerned with the construction and analysis of arrangements in order to observe symmetry and satisfy certain balance properties. These arrangements can involve any objects, but are usually regarding finite sets. For example, a sudoku is a combinatorial design since it represents the arrangement of nonzero digits in a square grid. The area of combinatorial design is highly developed, although many interesting problems and fundamental questions remain unsolved. Most of the methods for constructing designs rely on an algebraic structure called a *finite field* and the more general system of arithmetic.

Block design is a part of combinatorial design in which arrangements of points in subsets are used to model several things ranging from graph colouring to experimental design. Balanced incomplete block designs (BIBDs) are mathematical structures used in experimental design and combinatorial mathematics, particularly in the field of statistics. Experimental design itself connects applications in statistics to the theory of combinatorial mathematics.

The existence and properties of a balanced, symmetric block design are dependent on its parameters. Richard Wilson in the 1970s showed that the trivial necessary conditions for the existence of various kinds of designs are "asymptotically sufficient", which is a statistical way of looking at design theory. It essentially means that given k, "there exists a design for all but finitely many k satisfying these conditions.", where k is an integer that represents the number of finitely many items for which the design fails.

1.1. Definitions.

Definition 1.1. A *block design* is a structure that consists of two sets; a finite set of points or *varieties*, denoted by V, and another of subsets of V, denoted by $\mathcal{B}$. The cardinalities of V and $\mathcal{B}$ are represented by v and b respectively. Since each $B \in \mathcal{B}$ is a block, b the the number of blocks in the design. Note that b and v are considered *parameters* of the design.

Definition 1.2. A *regular* design is a design in which every point appears in the same number of blocks, which is denoted by the parameter r. For example, if the point 1 appears thrice in three different blocks, and this is true for all the other points in V, then $r = 3$ and the design is regular.

Definition 1.3. A *uniform design* is a design in which all the points are "uniformly" distributed among the blocks, i.e. all blocks have the same number of points, and this number is denoted by k.

Definition 1.4. A *balanced design* is a design in which every pair of distinct points in V appears together in the same number of blocks, λ.

Definition 1.5. A (b, v, k, r, λ) design is called a *complete* design if $v = k$, or it is simple and contains $\binom{v}{k}$ blocks. Otherwise, if $k = 1$, it is called a *trivial* design, since, there is only one point in every block. A regular, uniform balanced block design is also called a (b, v, r, k, λ) design.

Further, our primary objects of study, balanced incomplete block designs, are designs that are regular, uniform, balanced and incomplete, according to the definitions of the parameters above.

For example, a BIBD with parameters $(20, 16, 5, 4, 1)$ is

$$\{1,2,3,4\},\{5,6,7,8\},\{9,10,11,12\},\{13,14,15,16\}$$
$$\{1,8,12,16\},\{2,5,10,15\},\{2,6,9,16\},\{2,7,12,13\}$$
$$\{4,5,11,14\},\{4,6,12,13\},\{4,7,10,16\},\{4,8,9,15\}$$
$$\{1,5,9,13\},\{1,6,10,14\},\{1,7,11,15\},\{2,8,11,14\}$$
$$\{3,5,12,15\},\{3,6,11,16\},\{3,7,9,4\},\{3,8,10,13\}$$

Figure 1. A BIBD with parameters $(20, 16, 5, 4, 1)$.

2. Construction of designs

There are several methods for constructing BIBDs, a few of which are given below. Since the parameters b and r can be expressed in terms of v, k and λ for regular, balanced and uniform designs as we will be doing in the following section, we may omit that notation for now.

- We begin with a BIBD (v, k, λ). We then create t copies of each block for all blocks in the design. The result is a BIBD of the form $(v, k, t\lambda)$.
- We once again start with a pre-existing design, (v, k, λ), from sets $(V, \mathcal{B})$. We replace each block $B \in \mathcal{B}$ with its *complementary block*, $B^c = V \setminus B = \{B^c : B \in \mathcal{B}\}$. Then $(V, \{B^c : B \in \mathcal{B}\})$. The design constructed by this method is called the *complementary design* or *complement* of the original design, and can be denoted by $(V, \mathcal{B}^c)$.

Theorem 1. *The complement of a BIBD (v, k, λ) is a BIBD $(v, v-k, b-2r+\lambda)$.*

Proof. The complement of a design still has v points and b blocks. The corresponding block of the complementary design will have the $v-k$ elements of $V \setminus B$.

We need to count the number of times any given pair of points appear together in a block of the complementary BIBD. Two points appear together in a block of the complementary BIBD if and only if neither of them was in the respective block of the original BIBD. Each of the two points appeared in r blocks of the original BIBD, and they appear together in λ blocks of the original BIBD.

Now we use the principle of inclusion-exclusion to count the number of blocks in which at least one point appears; it is $r + r - \lambda = 2r - \lambda$. Therefore, the number of blocks in which neither of the two points appears is $b-(2r-\lambda) = b-2r+\lambda$, and thus b in the complementary design equals $b - 2r + \lambda$. Since this counting technique is independent of the choice of our two points, the complement is indeed a BIBD, as every pair of points appears together in some block $b - 2r + \lambda$ times. ■

3. Incidence matrices

Suppose we have a design on the vertex set $V = \{0, 1, 2, 3, 4, 5, 6\}$ as follows:

$$\{0,1,2\},\{0,3,4\},\{0,5,6\},\{1,3,5\},\{1,4,6\},\{2,3,6\},\{2,4,5\}.$$

These blocks shall be denoted by $B_1, B_2, B_3, \cdots, B_7$ respectively. Given these, there exists an *incidence relation* $\mathcal{R}$ between $V = \{01, 2, 3, 4, 5, 6\}$ and $\mathcal{B} = \{B_1, B_2, B_3, \cdots, B_7\}$. This incidence relation is defined by the point a_i, where the ordered pair $(a_i, B_j) \in \mathcal{R}$ if $a_i \in B_j$. We put a one in the (a_i, B_j) entry if $(a_i, B_j) \in \mathcal{R}$, and a zero otherwise. It is represented by the following table or matrix.

	B_1	B_2	B_3	B_4	B_5	B_6	B_7
0	1	1	1	0	0	0	0
1	1	0	0	1	1	0	0
2	1	0	0	0	0	1	1
3	0	1	0	1	0	1	0
4	0	1	0	0	1	0	1
5	0	0	1	1	0	0	1
6	0	0	1	0	1	1	0

Figure 2. Table of incidence $\mathcal{R}$

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Figure 3. Matrix of incidence $\mathcal{R}$

Incidence matrices help us describe the information in a BIBD in a concise manner. It is important to note here that the content of an incidence matrix can vary according to the manner in which we label the points and blocks in a BIBD. Therefore, while arranging the content in a BIBD into an incidence matrix, we must be very clear about the points and blocks.

4. Graph colouring

In this section, we discuss the alternate representation of BIBDs using graph colouring.

Definition 4.1. A *multigraph* is a graph that includes *loops*, i.e. vertices that connect edges to themselves, and *parallel edges*, i.e. multiple edges constructed between the same pair of vertices.

Definition 4.2. A *complete graph* is a graph in which every pair of distinct vertices is directly connected by a unique edge.

Note: Here K_v denotes a *simple, complete* graph K on v vertices. Similarly, K_k denotes a simple complete graph K on k vertices. The multigraph λK_v denotes the multigraph in which each edge of K_v has been replaced by λ copies of that edge.

Theorem 2. *Colouring the edges of the multigraph λK_v so that the edges of any given colour form a K_k results in the BIBD (v, k, λ).*

Proof. Let the points in the design represent the set of vertices in the multigraph λK_v. For each colour n that we use, we will create a block whose points are the vertices in K_k which also have the colour n. Since it is a uniform, balanced design, all blocks will have cardinality k.

Every vertex has degree $\lambda(v-1)$, and every complete graph on k vertices (K_k) of one colour containing that vertex will have $k-1$ edges incident with that vertex. Therefore, every vertex appears in

$$r = \frac{\lambda(v-1)}{(k-1)}$$

blocks. Since any edge of λK_v must appear in some K_k that is coloured with the colour of that edge, we may say that for any pair of points, these vertices are connected by λ edges each of which appears in some K_k. Therefore, these points may appear together in λ of the K_k subgraphs, or blocks.

We label the vertices of K_v with the points of the design, given a BIBD (v, k, λ) and the multigraph λK_v. For every block in the design, we use a new colour for each of the edges of a K_k that connects the points in that block. Since every pair of points appears together in exactly λ blocks, there will be sufficiently many uncoloured edges joining these points, and there will be λ edges joining the respective vertices. Therefore, this results in every edge of the multigraph being coloured. ■

4.1. **Block intersection graphs.** Another method of representing BIBDs are through *block intersection graphs*. Vertices of this graph correspond to the blocks of the design. Two vertices are adjacent if their respective blocks have a non-empty intersection. For example, a BIBD with parameters $(20, 16, 5, 4, 1)$ and its respective block intersection graph are as follows.

$\{1,2,3,4\}, \{5,6,7,8\}, \{9,10,11,12\}, \{13,14,15,16\}$
$\{1,8,12,16\}, \{2,5,10,15\}, \{2,6,9,16\}, \{2,7,12,13\}$
$\{4,5,11,14\}, \{4,6,12,13\}, \{4,7,10,16\}, \{4,8,9,15\}$
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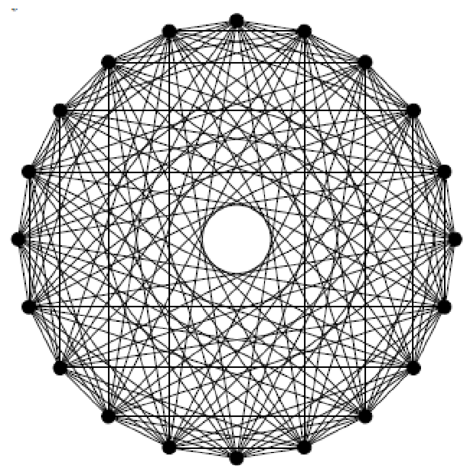
$\{3,5,12,15\}, \{3,6,11,16\}, \{3,7,9,4\}, \{3,8,10,13\}$



5. Conditions for the existence of a regular, uniform, balanced design

Theorem 3. *Two parametric conditions for the existence of a (b, v, r, k, λ) design are as follows:*

(1) $bk = vr$
(2) $r(k-1) = \lambda(v-1)$

We provide combinatorial proofs for these conditions that are based on counting the number of appearances of certain points.

Proof. There are b blocks and each has k points, so the total number of points in the design (counting repetitions as well) is bk, using the multiplication principle of counting. In the set

V there are v points, each of which appear r times. Both the LHS and RHS of the equation represent the total number of points in the design, so $bk = vr$.

For the second condition, consider a certain point x in the design. This point appears in r blocks. Besides x, each block has $k-1$ points, each of which are potential choices for the point x. So the total number of appearances of the point x is $r(k-1)$. Another way to count this is: for every point $y \in V$ such that $x \neq y$, the pair (x, y) appears in λ different blocks (this is only possible if the design is balanced). Since y can be any point among the remaining $v-1$ points, the number of appearances of x along with $v-1$ choices for y in a pair is $\lambda(v-1)$. ■

A design is either called a (v, k, λ) design or a (b, v, r, k, λ) design. Only regular, uniform and balanced designs are called (v, k, λ) designs, since given that they are regular, uniform and balanced, b and r can be derived directly from the values of v, k and λ. The method of derivation is explained in the proof of the following theorem.

Theorem 4. *A regular, balanced, uniform BIBD has*

$$\frac{\lambda v(v-1)}{k(k-1)}$$

blocks.

Proof. Since $bk = vr$ and $r(k-1) = \lambda(v-1)$, and $b = \frac{vr}{k}$ and $r = \frac{\lambda(v-1)}{k-1}$ result from rearrangement. Substituting the value of r from the first equation into the second gives

$$b = \frac{vr}{k} = \frac{v\left(\frac{\lambda(v-1)}{k-1}\right)}{k} = \frac{v}{k} \cdot \frac{\lambda(v-1)}{k-1} = \frac{\lambda v(v-1)}{k(k-1)}.$$

■

Since b and r are parameters that can be deduced from λ, v and k, as demonstrated above, a regular, uniform, balanced design can also be called a (v, k, λ) design.

Theorem 5. *For a* (v, k, λ) *BIBD,*

$$\lambda(v-1) \equiv 0 \pmod{k-1} \quad \textit{and} \quad \lambda v(v-1) \equiv 0 \pmod{k(k-1)}.$$

Proof. Note that b and r are compulsorily integers since they represent the number of blocks in different ways. We know from the parametric conditions of the existence of a BIBD that $\lambda(v-1) = r(k-1)$, we know that

$$\lambda(v-1) \equiv 0 \pmod{k-1}$$

. Since $b = \frac{\lambda v(v-1)}{k(k-1)}$, we know that

$$\lambda v(v-1) \equiv 0 \pmod{k(k-1)}$$

. ■

6. Parameter Relations

Theorem 6. *Suppose that $r \mid b$ in a (b, v, r, k, λ) design. Then*

$$b \geq v + r - 1.$$

Proof.

Lemma 6.1. $\dfrac{\lambda(n-1)}{(k-1)}$ *is an integer.*

Proof. Given that b is a divisor of r, we know that $b \equiv 0 \pmod{r}$ and thus $b = nr$ for some positive integer n. Since we know that one of the existence conditions for a BIBD is $\lambda(v-1) = r(k-1)$ as discussed earlier, we may say that

$$r = \frac{\lambda(v-1)}{(k-1)}.$$

We know from earlier that $bk = vr$, and thus substituting $b = nr$ into $bk = vr$ gives $vr = nrk$ or $v = nk$. This is an interesting observation, since this relation occurs between these parameters only when b divides r. Now we have a new substitution for v in

$$r = \frac{\lambda(nk-1)}{(k-1)}.$$

Since n and λ are positive integers, λn is also a positive integer. Since

$$r = \frac{\lambda(n-1)}{k-1} + \lambda n$$

must also be a positive integer since it is a parameter and counts a positive quantity, we know that

$$\frac{\lambda(n-1)}{(k-1)} \in \mathbb{Z}_{+}.$$

■

Assume for the sake of contradiction that $b < v + r - 1$. Substituting $b = nr$ in this inequality gives $nr < v + r - 1$. Rearranging and factoring gives $nr - r < v - 1$ or $r(n-1) < v - 1$.

From our parameter condition $\lambda(v-1) = r(k-1)$, we have $v - 1 = \frac{r(k-1)}{\lambda}$. Substituting this value of $v - 1$ into $r(n-1) < v - 1$ gives

$$r(n-1) < \frac{r(k-1)}{\lambda}$$

$$r(n-1) \cdot \frac{\lambda}{r(k-1)} < 1$$

$$\frac{\lambda(n-1)}{(k-1)} < 1$$

We thus arrive at a contradiction as the above expression is a positive integer, which cannot be less than one. Therefore our assumption that $b < v + r - 1$ is false and we may conclude that $b \geq v + r - 1$. ■

6.1. **Bruck-Ryser-Chowla Theorem.** This theorem was originally proposed by mathematicians R.H. Bruck, S. Chowla, and H.J. Ryser in [13].

Theorem 7 (Bruck-Ryser-Chowla Theorem)**.** *A symmetric* (v, k, λ) *BIBD must satisfy the following conditions:*

- *if* v *is even, then* $k - \lambda$ *is a square*
- *if* v *is odd, then the Diophantine equation*

$$x^2 - (k - \lambda)y^2 + (-1)^{\frac{v-1}{2}} \lambda z^2 = 0$$

has a nontrivial integer solution.

7. Fisher's Existence Theorem

Fisher's Existence Theorem is a fundamental result that provides a necessary condition for the existence of a (v, k, λ) block design. We present the statement and subsequent proof of Fisher's Existence Theorem. While this is essentially a major necessary condition for the existence of a BIBD, this condition has a significantly more elaborate proof, and there are other existence conditions that can directly derived from this theorem, which will be elaborated on later in this section as consequences of this theorem.

Theorem 8 (Fisher's Existence Theorem)**.** *Suppose there exists a BIBD* (v, k, λ)*. Then the following must hold:*

$$v \equiv b \equiv \lambda \equiv 0 \pmod{k-1}.$$

Proof. We begin by assuming the existence of a BIBD (v, k, λ). Let the number of blocks containing the point i as $r(i)$. Since the design is balanced and all points appear at least once in the same number of blocks, we have $r(1) = r(2) = r(3) = \cdots = r(v)$.

Since each block contains k points and the number of blocks is b, the total number of points across the design is bk. Therefore, we know that

$$\sum_{i=1}^{v} r(i) = bk. \tag{7.1}$$

Consider the number of pairs of the points occurring together in the blocks. Since each block contains k points, and the pair of points appears λ times across the design, the total number of pairs across all blocks is $b\lambda$. This can be expressed alternatively as

$$\frac{v(v-1)}{2} \times \lambda = b\lambda.$$

Simplifying this, we obtain

$$b = \frac{v(v-1)}{2}. \tag{7.2}$$

Consider any two points i and j. Each point appears $r(i)$ and $r(j)$ times respectively. Suppose the number of blocks containing both treatments is denoted by $r(i, j)$. Since each pair of treatments appears λ times, we obtain

$$r(i, j) \times \lambda = r(i) \cdot r(j). \tag{7.3}$$

Creating the summation of the above equation over all points in the point set gives

$$\sum(r(i,j)\cdot\lambda) = \sum(r(i)\cdot r(j)). \tag{7.4}$$

Expanding the summations gives

$$\lambda\cdot\sum r(i,j) = \sum r(i)\cdot\sum r(j). \tag{7.5}$$

We may substitute the value of $\sum r(i)$ and $\sum r(j)$ with bk, which results in

$$\sum r(i,j) = \frac{(bk)^2}{\lambda}. \tag{7.6}$$

Since $r(i,j) = b$ as defined earlier, we have

$$b = \frac{(bk)^2}{\lambda}.$$

Therefore

$$\lambda = \frac{(bk)^2}{b} = b\cdot k^2. \tag{7.7}$$

We can derive from existence conditions that

$$b\cdot k^2 \equiv 0 \pmod{(k-1)} \implies b\cdot k \equiv 0 \pmod{(k-1)} \implies b \equiv 0 \pmod{(k-1)}. \tag{7.8}$$

Similarly, we have

$$\lambda = b\cdot k^2 \equiv 0 \pmod{(k-1)}. \tag{7.9}$$

Therefore, we may conclude that $v \equiv 0 \pmod{(k-1)}, b \equiv 0 \pmod{(k-1)}$ and $\lambda \equiv 0 \pmod{(k-1)}$. ■

7.1. **Consequences of Fisher's Existence Theorem.**

7.1.1. *Bose's Theorem.* Bose's Theorem focuses specifically on the existence of resolvable BIBDs. As we will see in a later section, a resolvable BIBD is a BIBD whose blocks can be partitioned into sets, each of which is a partition of the point set.

Theorem 9. *Suppose there exists a resolvable BIBD* (v,k,λ). *Then the following must hold true:*

- $v \equiv 1 \pmod{k}$
- $b \equiv 0 \pmod{k}$

7.1.2. *Paley's Theorem.* Paley's Theorem is particularly useful in constructing BIBDs with specific block sizes based on prime powers congruent to 1 modulo 4.

Theorem 10. *Suppose there exists a BIBD* (v,k,λ) *where* k *is a prime power congruent to* 1 *modulo* 4. *Then the following conditions must hold:*

- $v \equiv 1 \pmod{k}$
- $b \equiv 1 \pmod{k}$
- $\lambda \equiv 1 \pmod{k}$

8. Fisher's inequality

Theorem 11. *Any BIBD (v, k, λ) must satisfy $b \geq v$.*

Proof. We know from Theorem 3 that

$$b = \frac{\lambda v(v-1)}{k(k-1)}.$$

So attempting to prove that $b \geq v$ would mean proving that

$$\frac{\lambda v(v-1)}{k(k-1)} \geq v.$$

Dividing both sides of the inequality by v gives

$$\frac{\lambda(v-1)}{k(k-1)} \geq 1.$$

Therefore, we can say that $b \geq v$ is equivalent to $\lambda(v-1) \geq k(k-1)$.
Let A be an arbitrary yet particular block of the BIBD (v, k, λ). For all values of i between 0 and k inclusive, suppose n_i denotes the number of distinct blocks A and A' such that the cardinality of the intersection of these two blocks is i.
Now we shall define a few equations using simple combinatorial statements.

$$\sum_{i=0}^{k} n_i = b - 1 \tag{8.1}$$

Note that the right hand side counts all the blocks excluding A, and the left hand side does the same thing, representing all blocks other than A in a summation of n_i.

$$\sum_{i=0}^{k} i n_i = r(k-1) \tag{8.2}$$

Note that the RHS and LHS both count the number of times that points present in A appear in a distinct block in the BIBD.

Counting the number of times all ordered pairs of points that are present in A appear in some other block in the design gives

$$\sum_{i=0}^{k} i(i-1)n_i = k(k-1)(\lambda-1). \tag{8.3}$$

Since $i(i-1)n_i = 0$ whenever $i = 0$ and $i = 1$, the index of the summation is actually 2 if we include these cases. Therefore, we can rewrite Equation 10.3 as

$$\sum_{i=0}^{k} i(i-1)n_i = \sum_{i=2}^{k} i(i-1)n_i = k(k-1)(\lambda-1).$$

The addition of equations 10.2 and 10.3 results in

$$\sum_{i=0}^{k} i^2 n_i = k(k-1)(\lambda-1) + k(r-1). \tag{8.4}$$

We switch perspective from this point and consider the polynomial in x given by

$$\sum_{i=0}^{k}(x-i)^2 n_i = \sum_{i=0}^{k}(x^2-2xi+i^2)n_i = x^2\sum_{i=0}^{k} n_i - 2x\sum_{i=0}^{k} in_i + \sum_{i=0}^{k} i^2 n_i.$$

Substituting values from Equations 10.1, 10.2, and 10.4 results in

$$x^2(b-1) - 2xk(r-1) + k(k-1)(\lambda-1) + k(r-1).$$

Since this polynomial originated from an expression that was the product of a sum of squares and nonnegative integers, its value must be nonnegative $\forall\ x \in \mathbb{R}$. Since the polynomial is a quadratic, its discriminant must be negative. This is due to the fact that if the quadratic is positive for all real values, then its graph is above the x-axis, and it does not have any real roots. Calculating its discriminant accordingly gives

$$(-2k(r-1))^2 - 4(b-1)(k(k-1)(\lambda-1) + k(r-1)) \leq 0.$$

Therefore, this can be simplified to

$$k^2(r-1)^2 - k(b-1)((k-1)(\lambda-1) + r - 1) \leq 0 \tag{8.5}$$

by factoring out k from the expression on the left hand side.

We know from earlier that $bk = vr$, so

$$k(b-1) = bk - k = vr - k.$$

Therefore, we may substitute $k(b-1)$ in Equation 10.5 with $vr-k$ to get

$$k^2(r-1)^2 - (vr-k)((k-1)(\lambda-1) + r - 1) \leq 0.$$

We partially expand the second term in the above equation to get

$$k^2(r-1)^2 - (vr-k)(k-1)(\lambda-1) - (vr-k)(r-1) \leq 0.$$

We multiply both sides of the inequality by $(v-1)$ to obtain

$$k^2(r-1)^2(v-1) - (vr-k)(k-1)(\lambda-1)(v-1) - (vr-k)(r-1)(v-1) \leq 0. \tag{8.6}$$

Note that the expression $(\lambda-1)(v-1)$ is present in the second term of the above equation. Since we know that $\lambda(v-1) = r(k-1) \implies \frac{\lambda(r(k-1))}{(v-1)}$, we have

$$\lambda = \frac{r(k-1)}{(v-1)} - 1 = \frac{r(k-1)-(v-1)}{(v-1)}.$$

Rearranging this gives

$$(\lambda - 1)(v - 1) = r(k - 1) - v + 1.$$

With our new substitution for $(\lambda - 1)(v - 1)$, our Equation 10.6 becomes

$$k^2(r-1)^2(v-1) - (vr-k)(k-1)(rk-r-v+1) - (vr-k)(r-1)(v-1) \leq 0.$$

Expanding and factoring this, we obtain

$$r(k-r)(v-k)^2 \leq 0.$$

Notice r is a positive integer, and $(v-k)^2$ is a perfect square so it must be nonnegative. Thus, $(k-r)$ must also be nonnegative, and subsequently we get $k \leq r$ and we can conclude that $\frac{r}{k} \geq 1$. Since $bk = vr$, we know that $b = \frac{vr}{k} \implies b = v \cdot \frac{r}{k}$, and since the ratio $\frac{r}{k}$ is greater than 1, we may subsequently conclude that $b \geq v$. ■

9. Symmetric BIBDs

Definition 9.1. A BIBD is called *symmetric* if $b = v$, i.e. its incidence matrix is a square matrix. It is often abbreviated as a SBIBD.

For example, the following collection

$$\{0,1,2\}, \{0,3,4\}, \{0,5,6\}, \{1,3,5\}, \{1,4,6\}, \{2,3,6\}, \{2,4,5\}$$

consists of subsets of $\{0,1,2,3,4,5,6\}$. It is an SBIBD with parameters $b = v = 7, k = r = 3, \lambda = 1$.

9.1. **Construction of an SBIBD.** We begin with a set $\mathbb{S}_v = \{0, 1, 2, \cdots, v-1\}$ and consider a $k-$subset $\mathcal{P}$ of $\mathbb{S}_v$. Then $\forall\, i \in \mathbb{S}_v$, the translate $i + \mathcal{P}$ is also a $k-$subset of $\mathbb{S}_v$. The subsets $\mathcal{P}, \mathcal{P}+1, \mathcal{P}+2, \cdots, \mathcal{P}+v-1$ are the blocks developed from $\mathcal{P}$ and the preexisting set $\mathcal{P}$ is the *starter block*. If the set $\mathcal{P}' = \{\mathcal{P}, \mathcal{P}+1, \mathcal{P}+2, \cdots, \mathcal{P}+v-1\}$ is a BIBD, then it is a symmetric BIBD with parameters $b = v, k = r$, and $\lambda = \frac{k(k-1)}{(v-1)}$.

10. t-designs

Definition 10.1. A $t-(v,k,\lambda)$ design is a design on v points with block cardinality k such that every t-subset of V appears in λ blocks.

Theorem 12. *In a t-design,* $\binom{k-i}{t-i} \Big| \lambda \binom{v-i}{t-i}$ $\forall\, i$ *such that* $0 \leq i \leq t-1$.

Proof. We know that there are $\binom{v}{t}$ t-subsets (subsets with cardinality t) from the v points in the design. Each appears in λ blocks, and therefore, $\lambda\binom{v}{t}$ t-subsets are present in total.

Now we shall consider the event where $i = 0$. In each of the b blocks, there are $\binom{k}{t}$ t-subsets in that block. Thus, across the whole design, there are $b\binom{k}{t}$ t-subsets present in total.

So we may conclude from this combinatorial proof that

$$b\binom{k}{t} = \lambda\binom{v}{t}.$$

If we consider any set of i points, there are $\binom{v-i}{t-i}$ t-subsets that include these points, and each such subset is present in λ blocks. For each of the blocks that contain these i points, we may complete our i-set and convert it to a t-set in $\binom{k-i}{t-i}$ ways. Therefore, we have counted such a block $\binom{k-i}{t-i}$ times in the previous combinatorial proof. We may conclude that $\binom{k-i}{t-i}$ is indeed a divisor of $\lambda\binom{v-i}{t-i}$, as required. ■

$$\begin{array}{cccccc}
\{1,5,6,10\}, & \{1,2,8,9\}, & \{2,3,6,7\}, & \{3,4,9,10\}, & \{4,5,7,8\}, & \{1,3,4,7\}, \\
\{2,4,5,10\}, & \{1,3,5,8\}, & \{1,2,4,6\}, & \{2,3,5,9\}, & \{4,6,8,9\}, & \{1,7,9,10\}, \\
\{3,6,8,9\}, & \{5,6,7,9\}, & \{2,7,8,10\}, & \{1,2,3,10\}, & \{1,2,5,7\}, & \{1,4,5,9\}, \\
\{1,3,6,9\}, & \{1,6,7,8\}, & \{1,4,8,10\}, & \{2,3,4,8\}, & \{2,4,7,9\}, & \{2,5,6,8\}, \\
\{2,6,9,10\}, & \{3,4,5,6\}, & \{3,5,7,10\}, & \{3,7,8,9\}, & \{4,6,7,10\}, & \{5,8,9,10\}
\end{array}$$

Figure 4. A 3-(10,4,1) design, $t = 3$

11. Steiner & Kirkman triple systems

This type of design traces back its origin to *Kirkman's schoolgirl problem*, proposed by Reverend Thomas Pennyngton Kirkman in 1847. It goes as follows:

> *In a boarding school, there are fifteen schoolgirls who always take their daily walk in rows of threes. How can it be arranged so that each schoolgirl walks in the same row with every other schoolgirl exactly once a week?*

We may think of the schoolgirls as individual points in a block design, and since they must travel in threes, there are always five blocks with three points in each block. If every schoolgirl walks in the same row with every other schoolgirl exactly once a week, then every possible pair of points must appear together in a block exactly once, which means $\lambda = 1$. This problem can be solved using a *Kirkman triple system*, which is elaborated on shortly.

Definition 11.1. A *triple system* is a regular, balanced design wherein all blocks have cardinality three. This is a BIBD with parameters $(v, 3, \lambda)$.

Definition 11.2. A *Steiner triple system* (STS) is a triple system with $\lambda = 1$.

A BIBD that is also a Steiner triple system is of the form $(v, 3, 1)$. Since there is a singular variable v in this system, we use $STS(v)$ to denote a Steiner triple system on v points.

Definition 11.3. A BIBD is *resolvable* if its blocks can be partitioned into sets, each of which is a partition of the point set. A resolvable STS is called a *Kirkman triple system*.

Theorem 13. *In a Steiner triple system,*

$$r = \frac{\lambda(v-1)}{2}$$

and

$$b = \frac{\lambda v(v-1)}{6}$$

.

Proof. The characteristic of a Steiner triple system is that $k = 3$, and since we already know that $\lambda(v-1) = r(k-1)$, rearranging and substituting $k = 3$ implies

$$r = \frac{\lambda(v-1)}{(k-1)} = \frac{\lambda(v-1)}{2}$$

.

Since $bk = vr$, we have

$$b = \frac{vr}{k} = \frac{\lambda v(v-1)}{k-1} = \frac{\lambda v(v-1)}{6}$$

. ■

Theorem 14 (Kirkman)**.** *There exists $STS(v)$ if and only if $v \equiv 1, 3 \pmod 6$.*

An STS is usually constructed using Latin squares, which are another type of combinatorial design. In order to prove the above condition for the existence of an STS(v), we require proofs of two main results on Latin squares.

11.1. **Latin Squares.** Latin squares are another type of combinatorial design, that somehow resemble a sudoku. They are most simply described as an $n \times n$ array.

Definition 11.4. A *Latin square* on a set of X points is an $n \times n$ array such that each point in X appears exactly once in each row and once in each column.

1	2	3	4
4	1	2	3
3	4	1	2
2	3	4	1

Figure 5. A 4×4 Latin square on the set $X = \{1, 2, 3, 4\}$.

Definition 11.5. A *symmetric Latin square* is a special Latin square in which the point in row i and column j is equal to the point in row j and column i, i.e. if this point is represented by A, a symmetric Latin square has $A_{(i,j)} = A_{(j,i)}$.

Lemma 11.6. *For every odd value of n we can construct a symmetric $n \times n$ Latin square with its elements appearing in order down the central diagonal (bottom right to top left).*

Proof. Let the entries of the first row in our Latin square be as follows:

$$1, \frac{n+3}{2}, 2, \frac{n+5}{2}, 3, \cdots, \frac{n-1}{2}, n, \frac{n+1}{2}.$$

Following this pattern, for all $i \geq 2$, the elements of row j will be the elements of row $j-1$ shifted to the left. Since all elements in a row are distinct according to our definition of a Latin square, and since it takes n shifts to the left in an $n \times n$ array to return to our original position, all the entries in any column are distinct.

0	1	2	3	4
1	2	3	4	0
2	3	4	0	1
3	4	0	1	2
4	0	1	2	3

Figure 6. Symmetric Latin Square of order 4

Note that the element in (p, q) moves to $(p+1, q+1) \pmod{n}$.

Therefore, the positions in which this element appears will be (x, y) for which $x+y \equiv p+q$ mod n. Since $i+j = j+i$, $(i, j) = (j, i)$ and we may conclude that the resulting Latin square is symmetric.

Similarly, we may argue that $(i, i) = (1, 2i-1 \pmod{n})$. However, since this is the exact position of i, we obtain the desired result. ■

Note: this argument is invalid when n is even, since that would imply the existence of non-integer values in some positions in the Latin square. However, we can adapt this result to whenever n is even as seen below.

1	7	2	8	3	9	4	10	5	11	6
7	2	8	3	9	4	10	5	11	6	1
2	8	3	9	4	10	5	11	6	1	7
8	3	9	4	10	5	11	6	1	7	2
3	9	4	10	5	11	6	1	7	2	8
9	4	10	5	11	6	1	7	2	8	3
4	10	5	11	6	1	7	2	8	3	9
10	5	11	6	1	7	2	8	3	9	4
5	11	6	1	7	2	8	3	9	4	10
11	6	1	7	2	8	3	9	4	10	5
6	1	7	2	8	3	9	4	10	5	11

Figure 7. A symmetric Latin square when $n = 11$ for an odd value of n.

Lemma 11.7. *For every even value of n there is a symmetric $n \times n$ Latin square with values $1, \cdots \frac{n}{2}, 1, \cdots, \frac{n}{2}$.*

Proof. Suppose the entries of the first row in our Latin square are as follows:

$$1, \frac{n+2}{2}, 2, \frac{n+4}{2}, 3, \cdots, \frac{n-2}{2}, n.$$

1	6	2	7	3	8	4	9	5	10
6	2	7	3	8	4	9	5	10	1
2	7	3	8	4	9	5	10	1	6
7	3	8	4	9	5	10	1	6	2
3	8	4	9	5	10	1	6	2	7
8	4	9	5	10	1	6	2	7	3
4	9	5	10	1	6	2	7	3	8
9	5	10	1	6	2	7	3	8	4
5	10	1	6	2	7	3	8	4	9
10	1	6	2	7	3	8	4	9	5

Figure 8. A symmetric 10x10 Latin square, for an even value of n

Once again, note that the elements in (i, j) are the same as that of $(i-1), j$ shifted to the left.

Similar arguments as those in the previous lemma prove the symmetry of this Latin square, since $(i, i) = (1, 2i - 1 \pmod{n})$. These are the elements $1, 2, \cdots \frac{n}{2}$ and since we know that

$$\frac{2(n+2j)}{2} - 1 \equiv 2j - 1 \pmod{n}$$

subsequently $(j, j) = (\frac{n+2j}{2}, \frac{n+2j}{2})$, each element is repeated in that order to obtain the desired result. ■

11.2. **Kirkman Problem.**

Kirkman. Necessary condition: Given that $\lambda = 1$ in a Steiner triple system, we know that v is odd since $v - 1 = 2r$ is even. Since $v(v-1) = 6b$ is a multiple of 6, we conclude that either $3 \mid v$ or $3 \mid (v-1)$.

Case 1: $3 \mid (v-1)$. Since $v - 1$ is even, then $v - 1$ is a multiple of 6, i.e., there is a nonnegative integer n that satisfies $v - 1 = 6n$. Thus $v = 6n + 1$.

Case 1: $3 \mid v$. Since v is odd, then v is 3 times an odd number. This means that $v = 3(2n+1)$ for any nonnegative integer n. Thus $v = 6n + 3$.

Sufficient condition: Suppose that $v \equiv 1, 3 \pmod 6$. We shall look at separate constructions of Steiner triple systems on v points, depending on whether $v \equiv 1 \pmod 6$ or $v \equiv 3 \pmod 6$. We aim to find colour classes for the edges of the complete graph on k vertices K_k such that each colour class contains K_3.

$v \equiv 3 \pmod 6$. Suppose v is of the form $6p + 3$. Label the vertices of K_{6p+3} with

$$u_1, \cdots, u_{2p+1}; v_1, \cdots, v_{2p+1};$$

and $w_1, \cdots w_{2p+1}$. From what we found earlier, we know that there exists a $((2q+1) \times (2q+1))$ Latin square since it exists for $1 \leq i \leq 2p + 1$. The element i appears in (i, i). For all

$1 \leq i, j \leq 2q+1$ such that $i \neq j$, if $(i,j) = l$ then we colour the edges that join the vertices in each of the following sets with a different colour:

$$\{u_i, u_j, v_l\}, \{v_i, v_j, w_l\}, \{w_i, w_j, u_l\}.$$

We find that (i,j) and (j,i) both give rise to the same colour classes due to the fact that our Latin square is symmetrical. Every edge of the form $u_i u_j, v_i v_j$, or $w_i w_j$ has been coloured, since we consider every pair $i \neq j$ and this must hold true for such an edge to exist. Since we are dealing with Latin squares, every possible entry l occurs somewhere in the row i. Therefore, every edge of the form $u_i v_l, v_i w_l$, or $w_i v_l$ has been coloured.

Now we consider the case where $i = j$. We know that $(i,i) = i$ if $i \leq q$ and $(i,i) = i - q$ if $i > q$.

Therefore, the only edges that are not yet coloured are of the form $u_i v_i, v_i w_i, w_i u_i$ when $i \leq q$ and $u_i v_{i-q}, v_i w_{i-q}$, and $w_i u_{i-q}$ when $i > q$. For all values i such that $1 \leq i \leq q$, we make the edges joining u_i, v_i, w_i into one colour class. Note that among the remaining edges that are not incident to a certain vertex x, every other vertex is at the receiving end of one of the edges. Therefore, if for every $q+1 \leq i \leq 2q$ we use new colours for the edges that join the vertices in the aforementioned sets, every edge incident with x will have been coloured. Since each colour class additionally forms a complete graph on three vertices K_3, we have constructed a Steiner triple system. ■

We use the above theorem to attempt to find the solution to the Kirkman Schoolgirl Problem by constructing an STS(15). We have $15 = 6(2) + 3$, thus $q = 2$. The points of our design are u_i, v_i and w_i for all i such that $1 \leq i \leq 2q+1 = 5$, so we require a symmetric 5×5 Latin square. The square is as follows: The blocks formed from this Latin square are

1	4	2	5	3
4	2	5	3	1
2	5	3	1	4
5	3	1	4	2
3	1	4	2	5

Figure 9. Symmetric 5x5 Latin square

as follows. Note that although the initial requirement of the Kirkman Schoolgirl Problem

$$\begin{array}{llllll}
\{u_1,u_2,v_4\}, & \{v_1,v_2,w_4\}, & \{w_1,w_2,u_4\}, & \{u_1,u_3,v_2\}, & \{v_1,v_3,w_2\}, & \{w_1,w_3,u_2\}, \\
\{u_1,u_4,v_5\}, & \{v_1,v_4,w_5\}, & \{w_1,w_4,u_5\}, & \{u_1,u_5,v_3\}, & \{v_1,v_5,w_3\}, & \{w_1,w_5,u_3\}, \\
\{u_2,u_3,v_5\}, & \{v_2,v_3,w_5\}, & \{w_2,w_3,u_5\}, & \{u_2,u_4,v_3\}, & \{v_2,v_4,w_3\}, & \{w_2,w_4,u_3\}, \\
\{u_2,u_5,v_1\}, & \{v_2,v_5,w_1\}, & \{w_2,w_5,u_1\}, & \{u_3,u_4,v_1\}, & \{v_3,v_4,w_1\}, & \{w_3,w_4,u_1\}, \\
\{u_3,u_5,v_4\}, & \{v_3,v_5,w_4\}, & \{w_3,w_5,u_4\}, & \{u_4,u_5,v_2\}, & \{v_4,v_5,w_2\}, & \{w_4,w_5,u_2\},
\end{array}$$
$$\{u_1,v_1,w_1\},\{u_2,v_2,w_2\},\{u_3,v_3,w_3\},\{u_4,v_4,w_4\},\{u_5,v_5,w_5\}$$

Figure 10. STS(15)

is to find an STS(15), it also has the additional condition that this design is resolvable into seven groups of five girls each, one for every day of the week.

We conclude with an important theorem regarding the existence of a Kirkman triple system, that is proven in [12].

11.3. **Ray-Chaudhuri & Wilson.**

Theorem 15. *There exists a Kirkman triple system whenever* $v \equiv 3 \pmod 6$.

12. Experimental Design

Definition 12.1. *Randomized block designs* refer to the arrangement of blocks of experimental units or subjects, proposed intially by Sir Ronald Fisher during his study of the design of experiments in agriculture.

Experimental design plays a crucial role in various scientific disciplines, enabling researchers to draw reliable conclusions from their studies. A balanced incomplete block design (BIBD) is a mathematical structure that provides a systematic approach to efficiently design experiments and analyze data. BIBDs have found extensive applications in diverse fields, including agriculture, biology, medicine, psychology, social sciences, and engineering.

The need for balanced incomplete block designs arises from the limitations of complete block designs and completely randomized designs. While complete block designs involve partitioning experimental units into homogeneous blocks and treating each block separately, they often require a large number of blocks, which can be impractical or costly. On the other hand, completely randomized designs lack the ability to account for potential sources of variability, leading to inefficient use of resources and reduced statistical power.

Definition 12.2. A *nuisance factor* is a factor that has a negative or adverse effect on the results of an experiment.

Nuisance factors may not be of primary interest but have potential to become statistical inaccuracies or sources of inaccurate experimental variation. They include factors such as the person who prepared the experiment, the time when the experiment was conducted, testing equipment, temperature of the room the experiment was conducted in, batches or quantities of raw material, and so on.

The usage of the word *block* is intended to describe the plots of agricultural land used for conducting experiments for fertilizers, fungicides and so on for varieties of crops. In agricultural studies, a *block* is defined as a set of *contiguous* plots of land under the assumption that nuisance factors such as fertility, moisture and climate are all identical.

Randomized Block Designs are used when experimental material is non homogeneous. It was first developed by R.A. Fisher in 1924. A unique characteristic of randomized blocking is that the number of blocks is equal to the number of blocked treatments. It is based on three principles of experimental design: replication, randomization, and local control. It is the most commonly used experimental design method in agriculture, and in the case of field experiments, the experimental material is divided among a number of equal blocks.

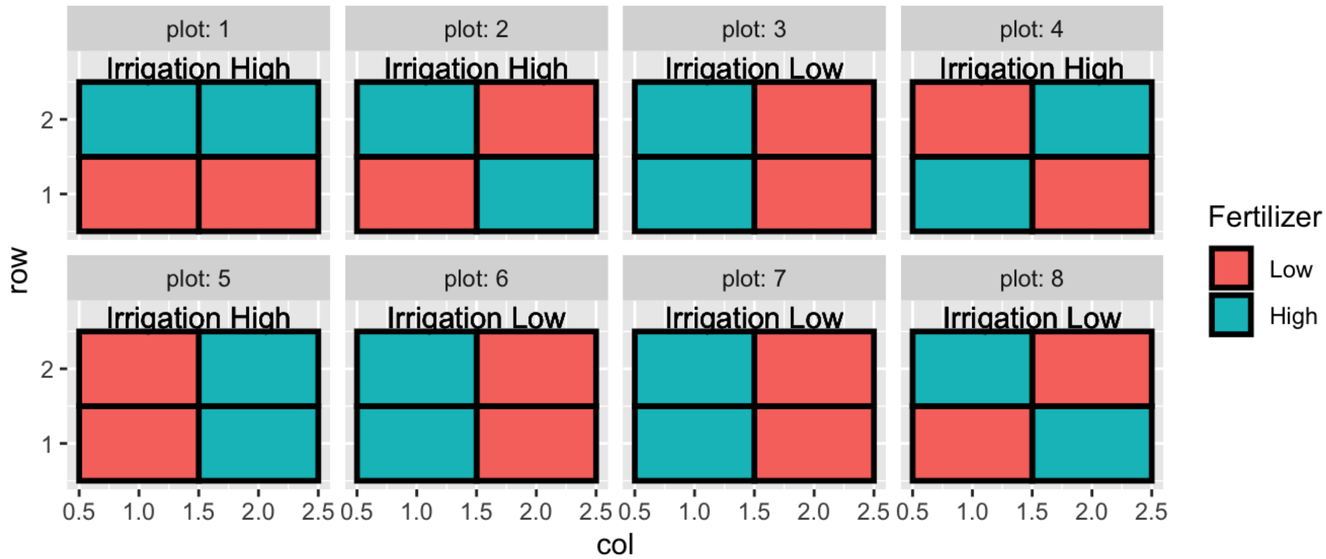


Figure 11. Agricultural field with varying irrigation and randomised blocking of fertilizer supply

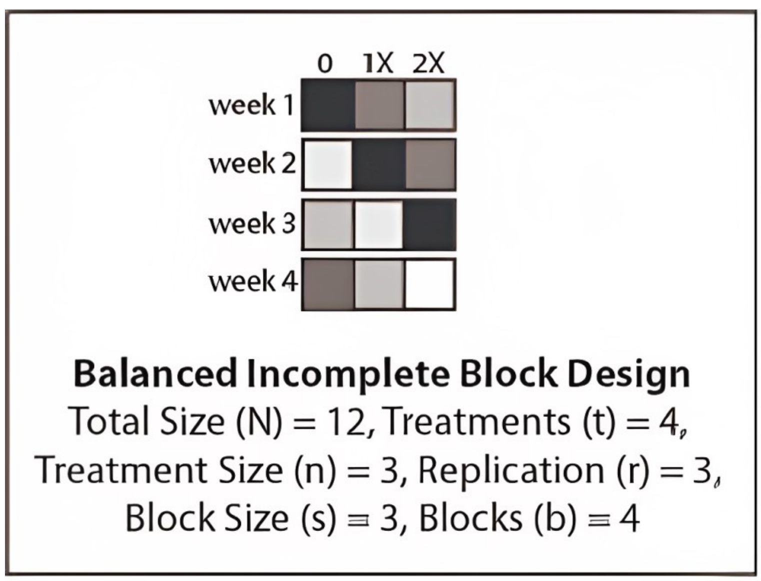


Figure 12. Blocking of treatments in a BIBD

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