Identifying sums of squares

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Basic modular arithmetic w/ squares

All squares have a remainder of either 0,1, or 4 when divided by 8. This can be deduced by either finding the square of each integer less than 8 and taking the remainder when dividing it by 8 or using induction(the difference between successive odd squares is always divisible by 8, for example).

Sums of two squares

Fermat's theorem: A non-negative integer is a sum of two squares if and only if every prime that is one less than a multiple of four in its prime factorization is raised to an even power.
For example. the integer, 29, is the sum of 25 and 4, both of which are squares, and 29, indeed, is a prime with only itself, which is 1 (mod 4) as a factor.

Sums of three squares

Legendre-Gauss theorem: A non-negative is a sum of three squares if and only if there do not exist any two integers, k and l, such that the non-negative integer is $4^k * (8l + 7)$.

Sums of three squares

For the "only if" part: From the slide on basic modular arithmetic, 0, 1, and 4 are the only possible remainders when a square is divided by 8. However, no combination of those 3 numbers adds up to 7 (mod 8).

Sums of three squares

In addition, if a number is one that is $7 \pmod 8$ times a power of 4, and a sum of 3 squares, they have to be divisible by 4, as if one was $1 \pmod 8$, their sum can't be $4 \pmod 8$, and thus, for any power of k that's a sum of three squares, the squares call be divided by 4, where they remain squares, and if $k > 1$, the quarters of the squares have to still be divisible by 4.

Sums of three squares

Thus, the number can be continually divided by 4, remaining a sum of three squares until it $\equiv 7 \pmod 8$, but it can't be a sum of three squares then, meaning it never could be.

Sums of four squares

Lagrange's theorem of four squares: Any non-negative integer is a sum of four squares, no matter what.
(Ex: $1179 = 33^2 + 9^2 + 3^2 + 0^2$, $3214 = 55^2 + 13^2 + 4^2 + 2^2$, etc.)

Sums of four squares

This can be proven from the last slide, as the a number that is 6 (mod 8) cannot ever be $4^k(8l+7)$, but any number that satisfies that $+\ 1$(which is a square) is a sum of three squares plus a square, and thus a sum of four squares, so any number that is $8l+7$ fits. In addition, any number that is a sum of three squares also fits, as 0 is a square.

Sums of four squares

If k is odd, meaning that a number $\equiv 4 \pmod 8$, the same conclusion applies. However, this isn't the case if a number is divisible by 8, but if a number is, it can simply be divided by 4 until it fits. Note that its largest factor not divisible by 4 has to fit one of the cases already proven.

Sums of increasing squares

The sum of $1^2 + 2^2 + 3^2 \ \ldots \ n^2$ is $\frac{n(n+1)(n+2)}{3} - \frac{(n+1)(n)}{2}$. (Ex. $1^2 + 2^2 + 3^3 ... 10^2 = \frac{(10)(11)(12)}{3} - \frac{(10)(11)}{(2)} = 440 - 55 = 385$)

Sums of increasing squares

The demonstration of this can be noted in that the difference of the difference between cubes increases incrementally by 6(i.e., $(2^3 - 1^3) - (1^3 - 0^3) = 6$, $(3^3 - 2^3) - (2^3 - 1^3) = 12$, $(4^3 - 3^3) - (3^3 - 2^3) = 18$, etc.

Sums of increasing squares

The proof of this is that the difference between x^3 and $(x+1)^3$ is $3x^2+3x+1$, and of $(x+1)^3$ and $(x+2)^3$ is $3x^2+9x+7$, making the difference between their differences $6x+6=6(x+1)$. This value incrementally increases by 6.

Sums of increasing squares

Therefore, a triangle can be constructed:
1
16
1,6,12
1, 6, 12, 18
Where a n^3 is simply the sum of all numbers in the triangle up to the nth row.

Sums of increasing squares

Removing the left-most coloumn converts the triangle into just a collection of multiples of 6s,

6

6, 12

6, 12, 18

Which can be divided by 3 to make it simply a triangle of even numbers, of which when each number is subtracted by 1, becomes

1

$1, 3$

$1, 3, 5$

where each row is just a square(squares increase by increasing odd numbers)!

Sums of increasing squares

Note that in the finished triangle's first row is the original triangle's 2nd row, meaning that the original triangle is $(n+1)^3$ rather than n^3 in the context where n corresponds to the finished triangle.
All the operations leave the final product as $1^2 + 2^2 + 3^2 \ldots\ n^2$ is $\frac{((n+1)^3-(n+1))}{3} - \frac{(n^2+n)}{2}$, which can be further converted into $\frac{n(n+1)(n+2)}{3} - \frac{(n+1)(n)}{2}$.

Thank you for listening