Weierstrass Approximation theorem

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Euler Circle

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Polynomials are special!

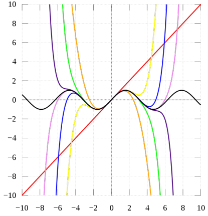


Figure 1: First few Taylor polynomials of $f(x) = \sin x$

Given $P(x) = \sum_{r=0}^{n} a_r x$ we can find the value of $P(x)$ at a point x using just addition and multiplication. Other functions are not as easy to compute, and that is why we introduce ways to approximate them by series of polynomials. The Taylor polynomials of a function can be used to approximate it for a given x, the accuracy of the estimation increasing as more terms are included. However, using the Taylor series to approximate a function requires it to be infinitely differentiable. The Weierstrass Approximation theorem, proved by Karl Weierstrass in 1885, is a'stronger' method of approximating a real-valued function on a closed interval as it only requires that the function be continuous.

Table of contents

Please note

Note that I will not be discussing the generalised form of the Weierstrass Approximation theorem (the Stone-Weierstrass theorem) in my talk. You can learn more in my paper.

Preliminaries

Definitions

Definition 1.1

(Convergent sequence in reals) A sequence $\{a_n\}$ in a metric space X is said to *converge* to a point a if for some $N \in \mathbb{N}$ and each $\epsilon > 0$,

$$n > N \implies |a_n - a| < \epsilon.$$

The point a is also called the *limit point* of $\{a_n\}$ and we write

$$a_n \to a.$$

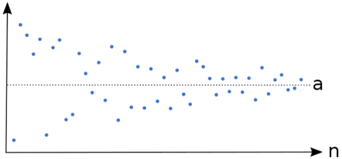


Figure 2: Convergent sequence

Definitions

Definition 1.2

(Uniform convergence in reals) A sequence of functions $f_n : [a, b] \to \mathbb{R}$ is said to converge uniformly to a limit function $f : [a, b] \to \mathbb{R}$ if for each $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ and for all $x \in [a, b]$,

$$|f_n(x) - f(x)| < \epsilon.$$

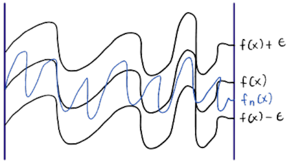


Figure 3: For large n, f_n is contained entirely in the ϵ-tube around f

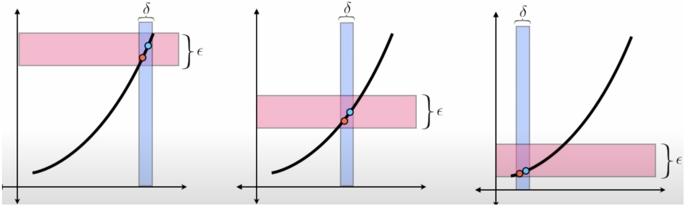
Definitions

Definition 1.3

(Continuity and uniform continuity) The function $f : [a, b] \to R$ is continuous if for each $\epsilon > 0$ and each $x, y \in X$ there exists a $\delta > 0$ such that

$$|y - x| < \delta \implies |f(x) - f(y)| < \epsilon.$$

A continuous function is said to be *uniformly continuous* if the choice of δ depends only on ϵ and not on x and y.



Definition 1.4

$(C([a, b], \mathbb{R})$ is defined as the set of all continuous functions from $[a, b] \to \mathbb{R}$.

Weierstrass Approximation theorem

Statement of the Weierstrass Approximation theorem

Theorem 2.1

(Weierstrass) Let $f \in C([a,b],\mathbb{R})$. Then there is a sequence of polynomials $p_n(x)$ that converges uniformly to $f(x)$ on $[a,b]$.

In other words, $\forall f \in C([a,b],\mathbb{R})$ and every $\epsilon > 0$, there is a polynomial function $p_n(x)$ such that $\forall x \in [a,b]$ and large enough n:

$$|f(x) - p_n(x)| < \epsilon.$$

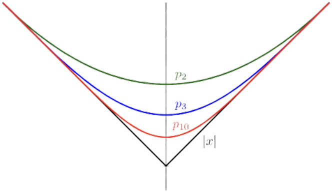


Figure 4: Sequence of polynomials converging uniformly to $f(x) = |x|$

Overview of Bernstein polynomials

Proof: Bernstein polynomials

- A constructive proof given by Sergei Bernstein in 1912
- A sequence of polynomials (Bernstein polynomials) that converge to the given function is explicitly defined

Theorem 3.1

(Bernstein) Define the Bernstein polynomial $B_n f(x)$ of a function $f(x)$ by

$$B_n f(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}$$

We have $B_n f \to f$ uniformly on $[0,1]$.

Remark 3.2

Proving the theorem for $[0,1]$ and using the mapping $\phi(x) = (b-a)x + a$ (where $x \in [0,1]$) will suffice to show that it holds for $[a,b]$.

Heuristic of Bernstein's proof

Coin-tossing game

Imagine a game where you toss a biased coin that lands heads with probability $x \in [0, 1]$. Each time you get k heads from n tosses, you receive an amount of money equal to $f(\frac{k}{n})$, where f is a continuous function. What is the expected value of the return on one play of the game as the number of tosses $n \to \infty$?

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- P(k heads from n tosses) $= \binom{n}{k} x^k (1-x)^{n-k}$.
- Expected value $E_n(x)$ of return on one play ($\sum$ return $\cdot$ probability of return):

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$$E_n(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}$$

- For large n, we expect the number of heads to be approximately equal to nx
- By the *Law of Large Numbers*, which states that the average of the results obtained from a large number of trials should be close to the expected value, we have:

$$\text{average return} = f(\tfrac{nx}{n}) = f(x) \approx \text{expected value for return} = E_n(x)$$

Sketch of proof

- Our main goal is to prove that for a function $f(x) : [0, 1] \to \mathbb{R}$, we have a sequence of polynomials $p_n(x)$ that approximates it really well on $[0, 1]$.

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- We fix a point $x_0 \in [0,1]$ and show that $f(x_0)$ is very close to $B_n f(x_0)$.
- Since x_0 was chosen arbitrarily, for each $x \in [0,1]$, we find that $B_n f(x) \to f(x)$ as $n \to \infty$.

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- We fix a point $x_0 \in [0,1]$ and show that $f(x_0)$ is very close to $B_n f(x_0)$.
- Since x_0 was chosen arbitrarily, for each $x \in [0,1]$, we find that $B_n f(x) \to f(x)$ as $n \to \infty$.
- Also, we will show that n is independent of x, which will help us establish the condition for uniform convergence.

Properties of Bernstein polynomials

$$B_n f(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}$$

Lemma 1

$$B_n(C) = C.$$

where C is a constant polynomial.

Lemma 2

$$B_n(f(x) - C) = B_n(f(x)) - C.$$

Lemma 3

$$h \geq g \implies B_n(h) \geq B_n(g).$$

Lemma 4

$$f \geq 0 \implies B_n(f) \geq 0.$$

Bernstein's proof explained

Cases

Goal

Find an upper bound for $|f(x) - f(x_0)|$ where x_0 is fixed in $[0, 1]$

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- Consider $x \in [0, 1]$.
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$$|x - x_0| \leq \delta \implies |f(x) - f(x_0)| \leq \epsilon$$

for some δ which depends on ϵ.

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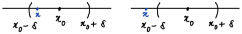
  for some δ which depends on ϵ.
- Fix an $\epsilon > 0$. We have 2 possibilities: $|x - x_0| \leq \delta$ and $|x - x_0| > \delta$.



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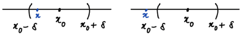

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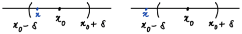

- The first case is accounted for by the definition of uniform continuity, the bound for $|f(x) - f(x_0)|$ being ϵ.
- We can focus on the case where $|x - x_0| > \delta$.

Finding a bound

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- Since f is continuous on $[0, 1]$, it is bounded. Take $M = sup\{f\}$.

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- Since f is continuous on $[0, 1]$, it is bounded. Take $M = sup\{f\}$.
- Use the Triangle Inequality on $|f(x) - f(x_0)|$

$$|f(x) - f(x_0)| \leq |f(x)| + |f(x_0)| \leq 2M.$$

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- Since $|x - x_0| > \delta$, $\frac{(x-x_0)^2}{\delta^2} > 1$.

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$$|f(x) - f(x_0)| \leq |f(x)| + |f(x_0)| \leq 2M < 2M\frac{(x - x_0)^2}{\delta^2}.$$

- Combining this with the case where $|x - x_0| \leq \delta$,

$$|f(x) - f(x_0)| < 2M\frac{(x - x_0)^2}{\delta^2} + \epsilon.$$

Back to Bernstein polynomials

$$B_n f(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}$$

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- By Lemma 3, we know that $h \geq g \implies B_n(h(x)) \geq B_n(g(x))$. Thus, $|B_n(f(x) - f(x_0))| \leq B_n(2M\frac{(x-x_0)^2}{\delta^2} + \epsilon)$.

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- Therefore,

$$|B_n(f(x) - f(x_0))| \leq \frac{2M}{\delta^2}B_n((x-x_0)^2) + \epsilon.$$

Simplifying the individual terms

$$B_n f(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}$$

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Find an expression for $B_n((x - x_0)^2)$

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- Using some combinatorial identities and factorisation, we obtain

$$\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}(\frac{k}{n}-x_0)^2 = (x-x_0)^2 + \frac{1}{n}(x-x^2).$$

Completing the proof

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- By Lemma 2, we obtain that $|B_n(f(x)) - f(x_0)| = |B_n(f(x) - f(x_0))|$ since $f(x_0)$ is a constant.
- Substitute the expression we obtained for $B_n((x - x_0)^2)$ in our bound for $|B_n(f(x) - f(x_0))|$.

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- Substitute $x = x_0$ and use $\max(x_0 - x_0^2) = \frac{1}{4}$ in

$$|B_n(f(x)) - f(x_0)| \leq \frac{2M}{\delta^2}(x - x_0)^2 + \frac{2M}{\delta^2 n}(x - x^2) + \epsilon.$$

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$$|B_n(f(x_0)) - f(x_0)| \leq \frac{M}{2\delta^2 n} + \epsilon.$$

Completing the proof

Goal

Find an upper bound for $|B_n(f(x_0)) - f(x_0)|$

- By Lemma 2, we obtain that $|B_n(f(x)) - f(x_0)| = |B_n(f(x) - f(x_0))|$ since $f(x_0)$ is a constant.
- Substitute the expression we obtained for $B_n((x - x_0)^2)$ in our bound for $|B_n(f(x) - f(x_0))|$.
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$$|B_n(f(x_0)) - f(x_0)| \leq \frac{M}{2\delta^2 n} + \epsilon.$$

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- Finally

$$|B_n f(x_0) - f(x_0)| < 2\epsilon.$$

Remark 4.3

This proves our theorem since ϵ is arbitrary.

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- We extended our bound for $|f(x) - f(x_0)|$ to find the maximum of $|B_n f(x) - f(x_0)|$.
- We took $x = x_0$ in our expression for the bound of $|B_n f(x) - f(x_0)|$, establishing that the distance between the a function and its corresponding Bernstein polynomial is arbitrarily small.

Thank you for your attention! You can learn more about generalisations and applications of the Weierstrass Approximation theorem from my paper.