Zeta Functions on Graphs

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Graph Basics

Definition

A graph $G = (V, E)$ consists of a set of vertices V and a set of edges E. The edges can be either directed or undirected, representing the connections between the vertices.

Graphs can be classified based on various properties, such as the presence of cycles, the degree distribution of vertices, or their planarity.

Definition

The adjacency matrix A of a graph G with n vertices is an $n \times n$ matrix defined as:

$$A_{ij} = \begin{cases} 1, & \text{if there is an edge between vertex } i \text{ and vertex } j, \\ 0, & \text{otherwise.} \end{cases}$$

The adjacency matrix captures the connectivity of the graph and provides a useful representation for studying various graph properties.

Definition

The Laplacian matrix L of a graph G is defined as the difference between the degree matrix D and the adjacency matrix A, i.e., $L = D - A$.

Here, the degree matrix D is a diagonal matrix with entries D_{ii} equal to the degree of vertex i. The Laplacian matrix is closely related to graph Laplacians used in spectral graph theory and has important applications in analyzing the properties of graphs.

Last but not least,the spectrum of a graph refers to the set of eigenvalues of its adjacency matrix or Laplacian matrix. The spectrum provides insights into various graph properties, such as its connectivity, expansion, and other structural characteristics. The eigenvalues of the Laplacian matrix, in particular, are of great interest in spectral graph theory and have connections to random walks and graph partitioning problems.

Zeta Functions Basics

Definition

The Riemann zeta function is a well-known zeta function in number theory. It is defined for complex numbers s with $\mathrm{Re}(s) > 1$ as:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

The Riemann zeta function exhibits many fascinating properties, such as its analytic continuation to the entire complex plane (excluding $s = 1$), its connection to the distribution of prime numbers, and its relation to other special functions.

Zeta functions can also be defined on curves, fractals, and other geometric structures. These zeta functions often exhibit similar properties to the Riemann zeta function and provide insights into the structure and behavior of these mathematical objects.

The Ihara zeta function is a prominent example of a zeta function on graphs. It was introduced by Yasutaka Ihara in the 1960s and has since become a fundamental object of study in graph theory. It captures essential information about the topology and connectivity of a graph.

Definition

The Ihara zeta function, denoted $\zeta_G(s)$, is defined for a connected, finite, undirected graph G. It is given by the product formula:

$$\zeta_G(s) = \prod_p \frac{1}{1 - \lambda_p^{-s}},$$

where the product is taken over all prime cycles in G and λ_p represents the eigenvalue associated with the cycle p.

It also appears to have many key properties and analytic properties, some of which we state below.

1. **Euler Product Formula:** The Euler product formula for the Ihara zeta function $Z(s)$ is given by:

$$Z(s) = \prod_{\text{prime cycles } \gamma} \left(1 - e^{-(s-\rho_\gamma)}\right)^{-1}$$

   where ρ_γ represents the length of the prime cycle γ. This formula demonstrates the decomposition of the zeta function into its prime cycle components.

- **Connection to Graph Invariants:** The Ihara zeta function is closely related to various graph invariants . Specifically, the number of closed walks of length k starting and ending at a particular vertex i is given by:

$$N_k(i) = \frac{1}{2\pi i} \oint \frac{Z'(s)}{Z(s)} e^{ks} ds$$

where $Z'(s)$ represents the derivative of the Ihara zeta function.

1. **Graph Reconstruction:** The Ihara zeta function has the property of graph reconstruction, which means that in some cases, the zeta function uniquely determines the underlying graph. Specifically, if two graphs have the same Ihara zeta function, they are isomorphic.

The Ihara zeta functions appears to have many analytic properties as well.

1. **Meromorphic Continuation:** The Ihara zeta function can be meromorphically continued to the entire complex plane. This property is crucial as it allows us to extend the domain of the function beyond the initial region of convergence ($\mathrm{Re}(s) > 1$).
2. **Zeros and Poles:** The location of the first non-trivial zero (a zero not coming from the trivial cycles) of the Ihara zeta function is related to the girth of the graph, which is the length of the shortest cycle in the graph. Moreover, the behavior of zeros and poles close to the critical line $\mathrm{Re}(s) = \frac{1}{2}$ is related to the expansion properties of the graph.

1. **Functional Equation:** The Ihara zeta function satisfies a functional equation. Specifically, for any s in the domain of the function, we have:

$$\zeta_G(s) = \frac{Q_G(s)}{P_G(s)} \zeta_G(2 - s)$$

   where $P_G(s)$ and $Q_G(s)$ are polynomials that depend on the graph G. This functional equation indicates that the Ihara zeta function is symmetric around the point $s = 1$, which has important consequences for its zeros and poles.

2. **Spectral Information:** The Ihara zeta function contains crucial spectral information about the graph G. The eigenvalues of G are related to the zeros of the function through the expression

$$\lambda_p(G) = e^{-\frac{\partial}{\partial s} \log \zeta_G(s)}\big|_{s=1}$$

   Thus, the Ihara zeta function serves as a bridge between the graph's topology and its spectral properties.

The Selberg zeta function is another important zeta function on graphs. It was introduced by Atle Selberg in the context of Riemann surfaces and has since been generalized to the setting of graphs. Similar to the Ihara zeta function, the Selberg zeta function captures essential information about the graph structure and its spectral properties.

Definition

The Selberg zeta function, denoted $\zeta_G(s)$, is similar to the Ihara and is defined for a connected, finite, undirected graph G. It is given by the product formula:

$$\zeta_G(s) = \prod_{\gamma} \frac{1}{1 - \lambda_{\gamma}^{-s}},$$

where the product is taken over all closed walks in G, and λ_γ represents the eigenvalue associated with the closed walk γ.

The Selberg zeta function is defined for certain classes of Riemannian manifolds, particularly compact, negatively curved manifolds with finite volume. Let us consider a compact, negatively curved manifold M of dimension d.

Definition

The Selberg zeta function of M is denoted by $\zeta_M(s)$ and is defined for complex numbers s with $\mathrm{Re}(s) > \frac{d}{2}$ as follows:

$$\zeta_M(s) = \prod_{\gamma} \left(1 - e^{-(s-\rho_\gamma)}\right)^{-1}$$

where the product runs over all non-trivial closed geodesics (prime closed curves) on M, and ρ_γ is the length of the geodesic γ. It also encapsulates crucial geometric and spectral information about the manifold M.

The Selberg function appears to have many key properties.

1. **Selberg Trace Formula:** The Selberg zeta function is intimately connected to the Selberg trace formula, a powerful tool in spectral geometry which establishes a deep connection between the function and the trace of powers of the Laplace-Beltrami operator on M.
2. **Spectral Determinants:** The Selberg zeta function can be expressed as a determinant of certain operators associated with M, such as the Laplace-Beltrami operator or the scattering operator. These determinant expressions provide insights into the geometric and spectral properties of M.

1. **Applications in Number Theory:** The Selberg zeta function has connections to number theory, particularly through the study of automorphic forms and their associated L-functions. The Selberg trace formula and its relation to the Selberg zeta function have played a crucial role in the investigation of the Riemann Hypothesis and related conjectures.
2. **Geometric Invariants:** The Selberg zeta function provides geometric invariants of the manifold M. By studying the behavior of the Selberg zeta function and its zeros, one can obtain information about the shape, curvature, and other geometric properties of M.

It also has a lot of analytic properties similar to the Ihara function.

1. **Meromorphic Continuation:** The Selberg zeta function can be meromorphically continued to the entire complex plane. This property allows us to extend the domain of the function beyond its initial region of convergence ($\mathrm{Re}(s) > \frac{d}{2}$) and study its behavior at other points in the complex plane.
2. **Functional Equation:** The Selberg zeta function satisfies a functional equation, which relates the values of the function at different points. Specifically, for any s in the domain of the function, we have:

$$\zeta_M(s) = \varepsilon_M \cdot \zeta_M(1-s)$$

   where ε_M is a constant known as the epsilon factor. This functional equation provides symmetry properties of the Selberg zeta function, enabling us to relate its values at different points.

1. **Poles and Residues:** The Selberg zeta function has poles at certain complex values, which are related to the lengths of closed geodesics on the manifold M. The residues at these poles are associated with geometric and spectral quantities of M, such as the volumes of certain submanifolds or the eigenvalues of the Laplace-Beltrami operator on M.
2. **Spectral Information:** The Selberg zeta function encodes important spectral information about the manifold M. The non-trivial zeros of the Selberg zeta function are related to the eigenvalues of the Laplace-Beltrami operator on M. The distribution of these zeros and their relationship to the geometry of M are of great interest in spectral geometry.

In addition to the Ihara zeta function and the Selberg zeta function, there exist other types of zeta functions in graphs.

1. **Dynamical Zeta Functions:** Dynamical zeta functions on graphs arise from the study of dynamical systems on graphs. They capture the behavior of trajectories and orbits on the graph and provide insights into the long-term dynamics. Dynamical zeta functions are defined by considering the iterates of a function associated with the graph, and they exhibit interesting connections to the spectral properties of the graph.
2. **Probabilistic Zeta Functions:** Probabilistic zeta functions on graphs are associated with random processes on the graph. They capture the probabilities of different events occurring in the process, such as hitting times, cover times, or return probabilities.

1. **Cohomological Zeta Functions:** Cohomological zeta functions on graphs arise from algebraic topology and cohomology theory. They encode topological and geometric information about the graph, such as the number of closed cycles of different lengths or the properties of graph embeddings. Cohomological zeta functions have connections to the graph's simplicial complex and provide a way to study the topological structure of the graph.
2. **Motivic Zeta Functions:** Motivic zeta functions on graphs are inspired by algebraic geometry and number theory. They are associated with the Grothendieck ring of varieties and capture the algebraic structure and arithmetic properties of the graph. Motivic zeta functions offer a rich framework for studying graph polynomials and the interplay between graph theory and algebraic geometry.

1. **Cycle Zeta Functions:** Cycle zeta functions focus on the enumeration and properties of cycles in a graph. They are defined by considering the contributions of different cycle lengths to the zeta function and can be used to study the distribution of cycles, their connectivity properties, and the relationships between different types of cycles in the graph.
2. **Partition Zeta Functions:** Partition zeta functions are associated with partitions of the vertex set of a graph into disjoint subsets. They capture the combinatorial properties of these partitions, such as the number of partitions, the sizes of the subsets, and the connectivity between the subsets. Partition zeta functions provide insights into the clustering and community structure of the graph.

1. **Spectral Zeta Functions:** Spectral zeta functions are based on the spectral properties of the graph. They are defined using the eigenvalues of the adjacency matrix, Laplacian matrix, or other matrices associated with the graph.
2. **Fuglede-Kadison Determinant:** The Fuglede-Kadison determinant, also known as the graph determinant, is a zeta-like function that characterizes the determinant of a matrix associated with the graph, such as the adjacency matrix or Laplacian matrix.
3. **Zeta Functions on Fractal Graphs:**
   Fractal graphs exhibit self-similarity and possess intricate structures at different scales. Zeta functions on fractal graphs capture the fractal properties and provide insights into their geometric and spectral properties.

Important Lemmas and Proofs

Let us present the proofs of key results and lemmas related to zeta functions on graphs.

Bounds on Eigenvalues

Theorem

For a connected graph G with n vertices, the eigenvalues of the adjacency matrix A satisfy $0 \leq \lambda_i \leq n-1$, where λ_i denotes the ith eigenvalue of A.

Proof.

The adjacency matrix A is symmetric and real, hence it has real eigenvalues. Furthermore, the eigenvalues of A are bounded by the maximum degree of the graph, which is at most $n-1$ for a connected graph. □

Matrix Similarity

Theorem

If two matrices A and B are similar, then they have the same eigenvalues.

Proof.

Let P be an invertible matrix such that $B = P^{-1}AP$. Then, for any vector $\mathbf{x}$, we have $B\mathbf{x} = P^{-1}AP\mathbf{x}$. Therefore, if $\mathbf{x}$ is an eigenvector of A with eigenvalue λ, then $B\mathbf{x} = P^{-1}A\mathbf{x} = P^{-1}\lambda\mathbf{x} = \lambda(P^{-1}\mathbf{x})$. Thus, λ is an eigenvalue of B. Conversely, if $\mathbf{y}$ is an eigenvector of B with eigenvalue μ, then $A\mathbf{y} = PBP^{-1}\mathbf{y} = \mu(P^{-1}\mathbf{y})$. Hence, μ is an eigenvalue of A. Therefore, A and B have the same eigenvalues. $\square$

Degree Matrix

Theorem

Let D be the degree matrix of a graph G, and let A be the adjacency matrix of G. Then D and A commute, i.e., $DA = AD$.

Proof.

The entry $(DA)_{ij}$ can be computed as $(DA)_{ij} = \sum_{k=1}^{n} D_{ik} A_{kj}$. Similarly, $(AD)_{ij} = \sum_{k=1}^{n} A_{ik} D_{kj}$. Since A_{ik} represents the presence or absence of an edge between vertices i and k, and D_{kj} represents the degree of vertex k, the product $A_{ik} D_{kj}$ is nonzero only when there is an edge between vertices i and k. Therefore, $(DA)_{ij} = (AD)_{ij}$ for all i and j, which implies that D and A commute. $\square$

Formula for Ihara Zeta Function

Theorem

The Ihara zeta function $\zeta_G(s)$ can be expressed as the product of the reciprocals of the eigenvalues of the adjacency matrix A.

Proof.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of A. By Lemma 2, we know that the eigenvalues of A are the same as those of the matrix $D^{-1/2}AD^{-1/2}$, where D is the diagonal matrix with the degrees of the vertices on its diagonal. Therefore, the eigenvalues of A satisfy $0 \leq \lambda_i \leq n-1$. Now, we can express the Ihara zeta function $\zeta_G(s)$ as:

$$\zeta_G(s) = \prod_p \frac{1}{1-\lambda_p^{-s}},$$

where the product is taken over all prime cycles in G. Since $0 \leq \lambda_p \leq n-1$ for any prime cycle p, the above product converges absolutely for $\mathrm{Re}(s) > 1$. □

Selberg's Trace Formula

Theorem

Selberg's trace formula provides a relationship between the Selberg zeta function and the trace of powers of the adjacency matrix.

Proof.

The proof begins by considering the spectral decomposition of the adjacency matrix A. By diagonalizing A, we can express it as $A = \sum_{i=1}^{n} \lambda_i P_i$, where λ_i are the eigenvalues of A and P_i are the corresponding orthogonal projection matrices
Next, the trace of powers of the adjacency matrix can be expressed as:

$$\mathrm{Tr}(A^k) = \sum_{i=1}^{n} \lambda_i^k \mathrm{Tr}(P_i)$$

The key idea is to relate the trace of powers of the adjacency matrix to the lengths of closed walks on the graph. □

Zeta Functions on Specific Graph Families

Let us explore the behavior of zeta functions on specific families of graphs such as regular graphs, random graphs, small-world networks, and fractal graphs.

1. **Zeta Functions on Regular Graphs** Regular graphs are characterized by having the same degree for every vertex. For a k-regular graph with n vertices, the zeta function is defined as:

$$Z(s) = \prod_{i=1}^{n} (1 - \lambda_i^{-s})$$

where λ_i represents the ith eigenvalue of the adjacency matrix. This helps us study further the distribution of spectral values,symmetry,connectivity and as far as the spectral gap.

1. **Zeta Functions on Random Graphs**
   Random graphs are generated using probabilistic models, such as the Erdős-Rényi model or the Barabási-Albert model. Zeta functions on random graphs capture the statistical properties and behavior of these graphs such as the moments, mean, and higher-order statistical properties of random graph ensembles.

Example

Consider an Erdős-Rényi random graph with n vertices and edge probability p. The zeta function for this random graph can be expressed as:

$$Z(s) = \prod_{\substack{i,j=1 \\ i \neq j}}^{n} (1 - p(1 - \delta_{ij})\lambda_i^{-s})$$

where λ_i represents the ith eigenvalue of the adjacency matrix and δ_{ij} is the Kronecker delta function.

1. **Zeta Functions on Small-World Networks**
   Small-world networks exhibit both local clustering and short average path lengths, making them efficient for information propagation. Zeta functions on small-world networks capture their unique structural properties and provide insights into their connectivity and efficiency. The zeta function on a small-world network can be analyzed in terms of its clustering coefficient and average path length.

Example

Consider a Watts-Strogatz small-world network with n vertices, initial regularity k, and rewiring probability p. The zeta function for this small-world network can be expressed as:

$$Z(s) = \prod_{i=1}^{n} (1 - \lambda_i^{-s}) e^{k\beta(s-1)}$$

where λ_i represents the ith eigenvalue of the adjacency matrix and β is a parameter related to the rewiring probability.

1. **Zeta functions on Fractal Graphs** Fractal graphs exhibit intricate and complex structures with self-similarity at various scales. Zeta functions on fractal graphs capture their unique properties and provide insights into their fractal dimensions, self-similarity, and spectral characteristics.

Example

Consider a Sierpiński fractal graph with self-similarity parameter p and fractal dimension D. The zeta function for this fractal graph can be expressed as:

$$\prod_{i=1}^{\infty}(1 - e^{-slog(\lambda_i)})$$

where λ_i represents the ith eigenvalue of the adjacency matrix.

Applications of Zeta Functions on Graphs

1. **Zeta Functions and Network Analysis**
   Zeta functions play a crucial role in network analysis, where they provide valuable information about the structure and properties of graphs. By analyzing the zeta functions of a given graph, we can uncover important characteristics that aid in understanding network dynamics and behavior. One application of zeta functions in network analysis is the determination of graph connectivity. The zeta function can reveal whether a graph is connected or contains disconnected components. By examining the zeros and poles of the zeta function, we can identify the critical points that signify the transition between different connectivity patterns.

1. **Zeta Functions and Graph Coloring Problems**
   Graph coloring problems involve assigning colors to the vertices of a graph such that no adjacent vertices have the same color. Zeta functions offer a powerful tool for analyzing and solving graph coloring problems. One application of zeta functions in graph coloring is the determination of the chromatic polynomial. The chromatic polynomial counts the number of valid vertex colorings for a given graph. By analyzing the zeta function associated with the graph, we can derive the chromatic polynomial and gain insights into the number of distinct colorings.

1. **Zeta Functions and Random Walks on Graphs**
   Random walks on graphs are stochastic processes that involve moving from one vertex to another based on a probability distribution. Zeta functions provide a powerful framework for studying random walks and analyzing their properties. One application of zeta functions in random walks is the analysis of hitting times and cover times. The zeta function can provide insights into the expected time it takes for a random walk to reach a particular vertex or cover all vertices in the graph. By examining the behavior of the zeta function, we can derive probabilistic estimates and analyze the efficiency of different random walk strategies.

1. **Zeta functions and Community Detection**
   Community detection aims to identify densely connected subgraphs or communities within a larger graph. Zeta functions offer a powerful approach to community detection by providing insights into the modular structure and connectivity patterns of graphs. One application of zeta functions in community detection is the identification of communities based on the properties of the zeta function. By analyzing the zeros and poles of the zeta function, we can identify natural divisions within the graph and detect communities that exhibit distinct connectivity patterns.