Chip-Firing Game on Connected Graphs

Adanur Nas

July 16, 2023

Introduction to Chip-Firing Game

Chip-Firing Game

1. a mathematical game that simulates the redistribution of resources/chips on a graph

Introduction to Chip-Firing Game

Chip-Firing Game

1. a mathematical game that simulates the redistribution of resources/chips on a graph
2. played with vertices, edges, and chips

Introduction to Chip-Firing Game

Chip-Firing Game

1. a mathematical game that simulates the redistribution of resources/chips on a graph
2. played with vertices, edges, and chips
3. played on a connected graph

Introduction to Chip-Firing Game

Chip-Firing Game

1. a mathematical game that simulates the redistribution of resources/chips on a graph
2. played with vertices, edges, and chips
3. played on a connected graph
4. can be played infinitely until all vertices have less chips than their degrees

Introduction to Chip-Firing Game

Chip-Firing Game

1. a mathematical game that simulates the redistribution of resources/chips on a graph
2. played with vertices, edges, and chips
3. played on a connected graph
4. can be played infinitely until all vertices have less chips than their degrees
5. under topics algebraic geometry, graph theory, combinatorial games, and discrete mathematics

Introduction to Chip-Firing Game

Chip-Firing Game

1. a mathematical game that simulates the redistribution of resources/chips on a graph
2. played with vertices, edges, and chips
3. played on a connected graph
4. can be played infinitely until all vertices have less chips than their degrees
5. under topics algebraic geometry, graph theory, combinatorial games, and discrete mathematics
6. only non-negative integer values can be assigned to vertices

Introduction to Chip-Firing Game

Definition

Vertex is a node or a point in a graph.

Introduction to Chip-Firing Game

Definition

Vertex is a node or a point in a graph.

Definition

Edge is a connection or link between two vertices.

Introduction to Chip-Firing Game

Definition

Vertex is a node or a point in a graph.

Definition

Edge is a connection or link between two vertices.

Definition

Degree is the number of edges connected to that vertex.

Introduction to Chip-Firing Game

Definition

Vertex is a node or a point in a graph.

Definition

Edge is a connection or link between two vertices.

Definition

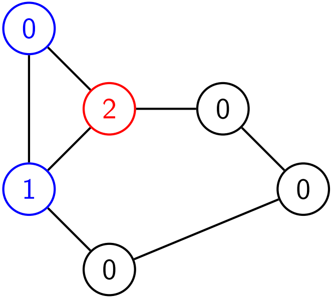
Degree is the number of edges connected to that vertex.

Definition

Configuration is the assignment of integer values to the vertices of a graph.

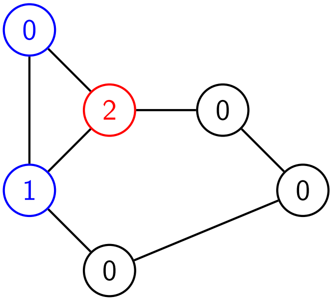
Introduction to Chip-Firing Game

The Game



Introduction to Chip-Firing Game

The Game



Red-colored vertex fires 1 chip each to blue-colored vertices.

Introduction to Chip-Firing Game

Theorem

In a *Chip-Firing Game* labeled with G, if there are n number of vertices and m number of chips to redistribute, then there are

$$v = \binom{m+n-1}{n-1}$$

ways to redistribute the chips.

Introduction to Chip-Firing Game

Proof

1. Consider a *Chip-Firing Game* labeled with G, consisting of n number of vertices and m number of identical chips to redistribute.

Introduction to Chip-Firing Game

Proof

1. Consider a *Chip-Firing Game* labeled with G, consisting of n number of vertices and m number of identical chips to redistribute.
2. Place *dividers* (representing chips) between *vertices*.

Introduction to Chip-Firing Game

Proof

1. Consider a *Chip-Firing Game* labeled with G, consisting of n number of vertices and m number of identical chips to redistribute.
2. Place *dividers* (representing chips) between *vertices*.
3. The number of *dividers* equals the number of *edges*.

Introduction to Chip-Firing Game

Proof

1. Consider a *Chip-Firing Game* labeled with G, consisting of n number of vertices and m number of identical chips to redistribute.
2. Place *dividers* (representing chips) between *vertices*.
3. The number of *dividers* equals the number of *edges*.
4. *Dividers* divide *vertices* into distinct regions, representing the chips in each *vertex*.

Introduction to Chip-Firing Game

Proof

1. Consider a *Chip-Firing Game* labeled with G, consisting of n number of vertices and m number of identical chips to redistribute.
2. Place *dividers* (representing chips) between *vertices*.
3. The number of *dividers* equals the number of *edges*.
4. *Dividers* divide *vertices* into distinct regions, representing the chips in each *vertex*.
5. Total regions $= n - 1$ (n vertices and $n - 1$ dividers).

Introduction to Chip-Firing Game

Proof

1. Consider a *Chip-Firing Game* labeled with G, consisting of n number of vertices and m number of identical chips to redistribute.
2. Place *dividers* (representing chips) between *vertices*.
3. The number of *dividers* equals the number of *edges*.
4. *Dividers* divide *vertices* into distinct regions, representing the chips in each *vertex*.
5. Total regions $= n - 1$ (n vertices and $n - 1$ dividers).
6. The problem is to determine combinatorial possibilities by selecting $n - 1$ dividers from $m + n - 1$ positions. Or,

Proof

1. Consider a *Chip-Firing Game* labeled with G, consisting of n number of vertices and m number of identical chips to redistribute.
2. Place *dividers* (representing chips) between *vertices*.
3. The number of *dividers* equals the number of *edges*.
4. *Dividers* divide *vertices* into distinct regions, representing the chips in each *vertex*.
5. Total regions $= n - 1$ (n vertices and $n - 1$ dividers).
6. The problem is to determine combinatorial possibilities by selecting $n - 1$ dividers from $m + n - 1$ positions. Or,

$$v = \binom{m+n-1}{n-1}$$

Diffusion

Definition

In *diffusion*, the chips are redistributed evenly and simultaneously among neighboring *vertices*.

Diffusion

Definition

In *diffusion*, the chips are redistributed evenly and simultaneously among neighboring *vertices*.

1. One of the most common *configuration* ways

Diffusion

Definition

In *diffusion*, the chips are redistributed evenly and simultaneously among neighboring *vertices*.

1. One of the most common *configuration* ways

2. At each step, each vertex sends a chip to each of its poorer neighbors.

Diffusion

Definition

In *diffusion*, the chips are redistributed evenly and simultaneously among neighboring *vertices*.

1. One of the most common *configuration* ways

2. At each step, each vertex sends a chip to each of its poorer neighbors.

3. In order to maintain consistency, the values of *vertices* can be negative, although such a thing is not possible in real life.

Diffusion

Definition

In *diffusion*, the chips are redistributed evenly and simultaneously among neighboring *vertices*.

1. One of the most common *configuration* ways

2. At each step, each vertex sends a chip to each of its poorer neighbors.

3. In order to maintain consistency, the values of *vertices* can be negative, although such a thing is not possible in real life.

4. Essential to understand *Pre-Positions*

Pre-Positions

Definition

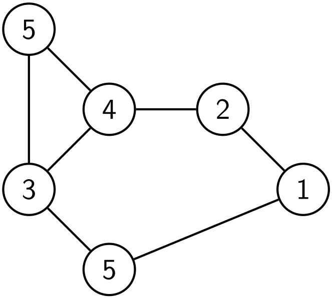
The previous configuration sequences obtained when a *Chip-Firing Game* is played backwards.

Pre-Positions

Definition

The previous configuration sequences obtained when a *Chip-Firing Game* is played backwards.

Example

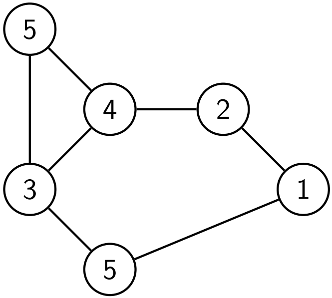


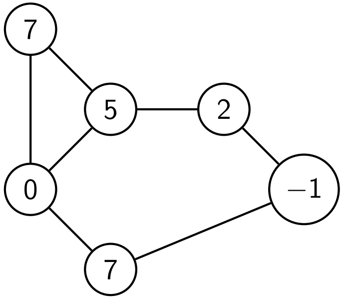
Pre-Positions

Definition

The previous configuration sequences obtained when a *Chip-Firing Game* is played backwards.

Example





Periods

Definition

A representation of the smallest positive integer such that after repeating the configuration sequence that many times, the configuration of chips returns to its original state

Periods

Definition

A representation of the smallest positive integer such that after repeating the configuration sequence that many times, the configuration of chips returns to its original state

1. This property refers to the repetition of the configuration sequence, not the firing sequence itself.

Periods

Definition

A representation of the smallest positive integer such that after repeating the configuration sequence that many times, the configuration of chips returns to its original state

1. This property refers to the repetition of the configuration sequence, not the firing sequence itself.
2. How many unique configurations, or *positions*, we need to identify before we reach the *period* depends on the initial configuration of our *Chip-Firing Game*.

Periods

Definition

A representation of the smallest positive integer such that after repeating the configuration sequence that many times, the configuration of chips returns to its original state

1. This property refers to the repetition of the configuration sequence, not the firing sequence itself.
2. How many unique configurations, or *positions*, we need to identify before we reach the *period* depends on the initial configuration of our *Chip-Firing Game*.
3. Not only *positions*, but *pre-positions* can also be used to find *periods* by using *diffusion*.

Periods

Definition

A representation of the smallest positive integer such that after repeating the configuration sequence that many times, the configuration of chips returns to its original state

1. This property refers to the repetition of the configuration sequence, not the firing sequence itself.
2. How many unique configurations, or *positions*, we need to identify before we reach the *period* depends on the initial configuration of our *Chip-Firing Game*.
3. Not only *positions*, but *pre-positions* can also be used to find *periods* by using *diffusion*.
4. *Periods* can be used to evidence that there are infinitely many unique *pre-positions* to any *Chip-Firing Game*.

Properties of Chip-Firing Game

1. *Chip-Firing Game* has been investigated and expanded by mathematicians, physicists, and economists since 1983.

Properties of Chip-Firing Game

1. *Chip-Firing Game* has been investigated and expanded by mathematicians, physicists, and economists since 1983.
2. The concept of placing chips on *vertices* can be modified to investigate energy distributions, revenue calculations, and more.

Properties of Chip-Firing Game

1. *Chip-Firing Game* has been investigated and expanded by mathematicians, physicists, and economists since 1983.
2. The concept of placing chips on *vertices* can be modified to investigate energy distributions, revenue calculations, and more.
3. Chips can represent various factors being analyzed, such as one chip representing the $100 profit made by selling 10 hats.

Properties of Chip-Firing Game

1. *Chip-Firing Game* has been investigated and expanded by mathematicians, physicists, and economists since 1983.
2. The concept of placing chips on *vertices* can be modified to investigate energy distributions, revenue calculations, and more.
3. Chips can represent various factors being analyzed, such as one chip representing the $100 profit made by selling 10 hats.
4. The rule of not assigning negative values can be extended.

Stability of Chip-Firing Game.

Definition

Analysis of how the distribution of chips or resources evolves over time and whether it reaches a stable configuration or continues to fluctuate indefinitely.

Stability of Chip-Firing Game.

Definition

Analysis of how the distribution of chips or resources evolves over time and whether it reaches a stable configuration or continues to fluctuate indefinitely.

1. Crucial for studying dynamics and long-term behavior of network structures and systems.

Stability of Chip-Firing Game.

Definition

Analysis of how the distribution of chips or resources evolves over time and whether it reaches a stable configuration or continues to fluctuate indefinitely.

1. Crucial for studying dynamics and long-term behavior of network structures and systems.
2. Important in *graph theory*, *network science*, and *distributed computing*.

Stability of Chip-Firing Game.

Definition

Analysis of how the distribution of chips or resources evolves over time and whether it reaches a stable configuration or continues to fluctuate indefinitely.

1. Crucial for studying dynamics and long-term behavior of network structures and systems.
2. Important in *graph theory*, *network science*, and *distributed computing*.
3. Helps analyzing the stability and resilience of complex networks like social networks, communication networks, and transportation networks.

Stability of Chip-Firing Game.

Definition

Analysis of how the distribution of chips or resources evolves over time and whether it reaches a stable configuration or continues to fluctuate indefinitely.

1. Crucial for studying dynamics and long-term behavior of network structures and systems.
2. Important in *graph theory*, *network science*, and *distributed computing*.
3. Helps analyzing the stability and resilience of complex networks like social networks, communication networks, and transportation networks.
4. Studying stability properties provides insights into the propagation and spread of information, resources, or influence within a network.

Stable States

Definition

The state in which no further firing is possible.

Stable States

Definition

The state in which no further firing is possible.

1. Also often referred to as a *firing-squad* or *firing-free configuration*.

Stable States

Definition

The state in which no further firing is possible.

1. Also often referred to as a *firing-squad* or *firing-free configuration*.
2. Occurs when no *vertex* in the network has a chip count that exceeds its outgoing *degree*. That is, each *degree* possesses enough chips to redistribute to its neighboring *vertices*

Stable States

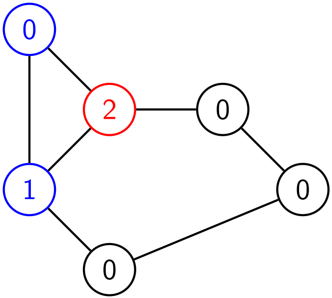
Definition

The state in which no further firing is possible.

1. Also often referred to as a *firing-squad* or *firing-free configuration*.
2. Occurs when no *vertex* in the network has a chip count that exceeds its outgoing *degree*. That is, each *degree* possesses enough chips to redistribute to its neighboring *vertices*
3. A balance in the system

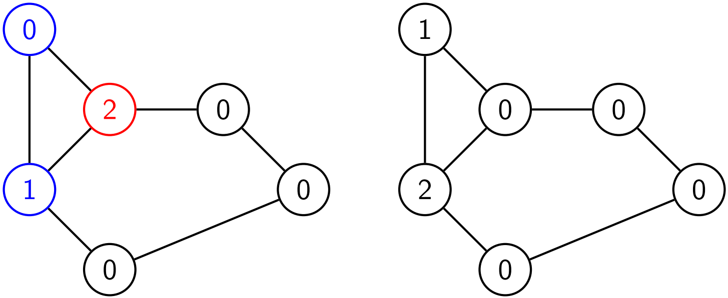
Stable States

Example



Stable States

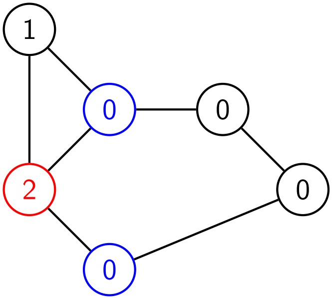
Example



Unlike some of the previous examples, we will not use diffusion in this example. Instead, we will fire red-colored vertices to blue-colored vertices.

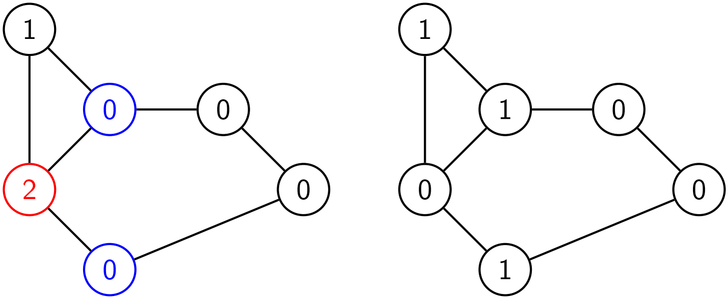
Stable States

Example - Continuation



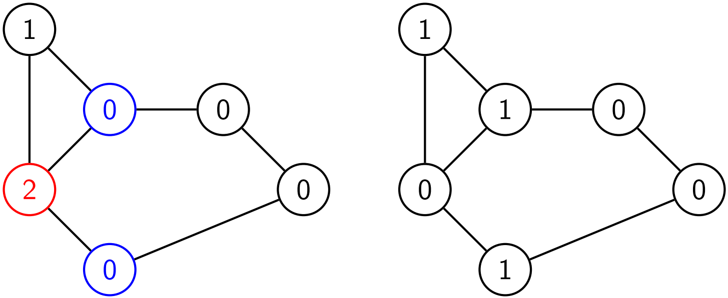
Stable States

Example - Continuation



Stable States

Example - Continuation



The firing has stopped, and the configuration has reached to a *stable state* because no *vertices* has equal amount of or more chips than their *degrees*.

Instability of Chip-Firing Game

This is also a crucial topic to further analyze in *Chip-Firing Game* even though many researchers focus on the concept of stability and *stable states*. This topic focuses on *unstable states*.

Instability of Chip-Firing Game

This is also a crucial topic to further analyze in *Chip-Firing Game* even though many researchers focus on the concept of stability and *stable states*. This topic focuses on *unstable states*.

Definition

A configuration, or state, in which further firing is possible and the chip distribution does not reach a balanced or stable state.

Instability of Chip-Firing Game

This is also a crucial topic to further analyze in *Chip-Firing Game* even though many researchers focus on the concept of stability and *stable states*. This topic focuses on *unstable states*.

Definition

A configuration, or state, in which further firing is possible and the chip distribution does not reach a balanced or stable state.

1. In an *unstable state*, certain *vertices* may accumulate an excessive number of chips, exceeding their *degrees*.

Instability of Chip-Firing Game

This is also a crucial topic to further analyze in *Chip-Firing Game* even though many researchers focus on the concept of stability and *stable states*. This topic focuses on *unstable states*.

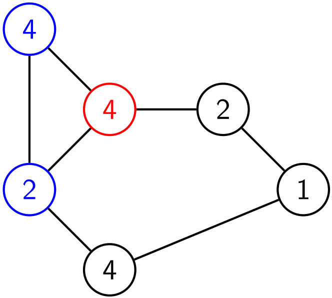
Definition

A configuration, or state, in which further firing is possible and the chip distribution does not reach a balanced or stable state.

1. In an *unstable state*, certain *vertices* may accumulate an excessive number of chips, exceeding their *degrees*.
2. *Unstable states* indicate a lack of equilibrium, and they can exhibit unpredictable chip dynamics, making it challenging to determine the long-term behavior of the system.

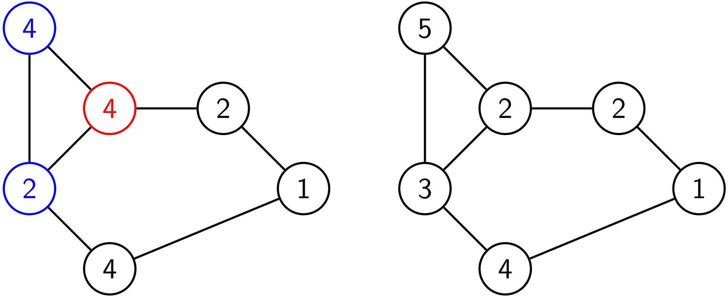
Unstable States

Example



Unstable States

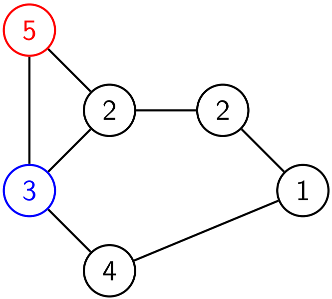
Example



Red-colored vertices will fire to blue-colored vertices.

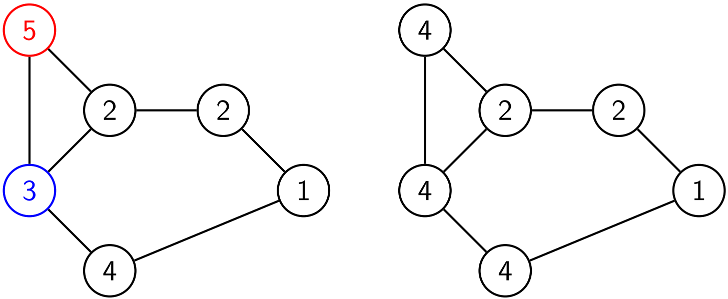
Unstable States

Example - Continuation



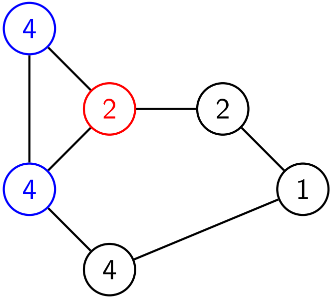
Unstable States

Example - Continuation



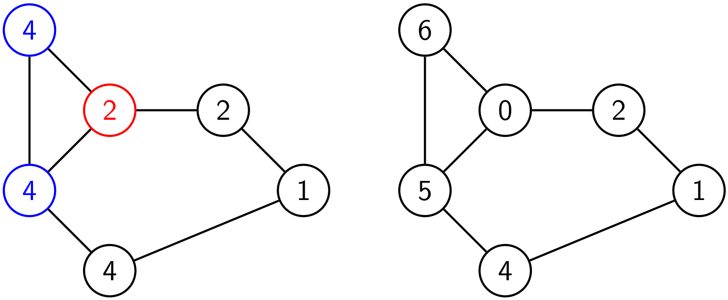
Unstable States

Example - Continuation



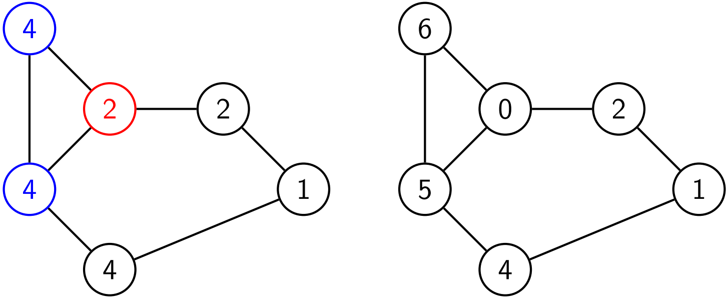
Unstable States

Example - Continuation



Unstable States

Example - Continuation



The last firing resulted in an *unstable state* because a *vertex* has 0 chips and a *degree* of 2, while its neighboring *vertices* have more chips than their respective *degrees*. Thus, the chip configuration will continue to fluctuate indefinitely.

Cyclic Configurations

Not all firing sequences follow a random pattern since some of them, while still maintaining an *unstable state*, can have a *cyclic configuration*.

Cyclic Configurations

Not all firing sequences follow a random pattern since some of them, while still maintaining an *unstable state*, can have a *cyclic configuration*.

Definition

A configuration that occurs when each *vertex* has at least the number of chips as their *degrees*, resulting in an infinite loop of firings and not allowing the game to reach a stable state.

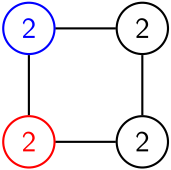
Cyclic Configurations

Not all firing sequences follow a random pattern since some of them, while still maintaining an *unstable state*, can have a *cyclic configuration*.

Definition

A configuration that occurs when each *vertex* has at least the number of chips as their *degrees*, resulting in an infinite loop of firings and not allowing the game to reach a stable state.

Example



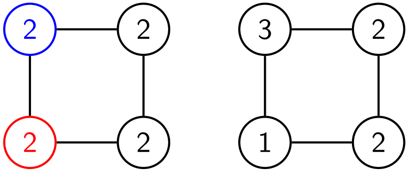
Cyclic Configurations

Not all firing sequences follow a random pattern since some of them, while still maintaining an *unstable state*, can have a *cyclic configuration*.

Definition

A configuration that occurs when each *vertex* has at least the number of chips as their *degrees*, resulting in an infinite loop of firings and not allowing the game to reach a stable state.

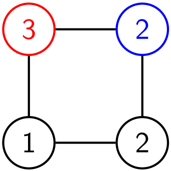
Example



Once again, red-colored vertices will fire to blue-colored vertices.

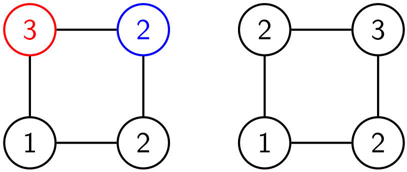
Cyclic Configurations

Example - Continuation



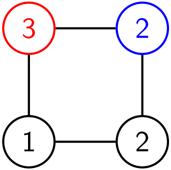
Cyclic Configurations

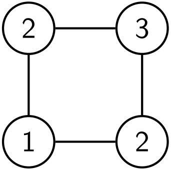
Example - Continuation



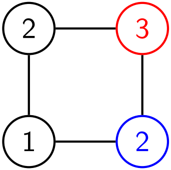
Cyclic Configurations

Example - Continuation



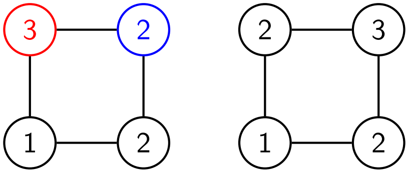


Example - Continuation

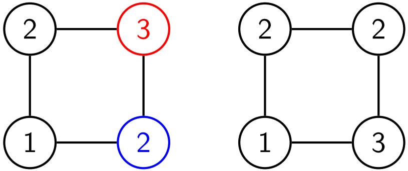


Cyclic Configurations

Example - Continuation

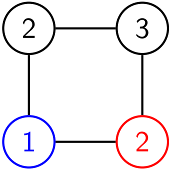


Example - Continuation



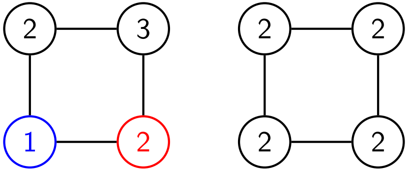
Cyclic Configurations

Example - Continuation



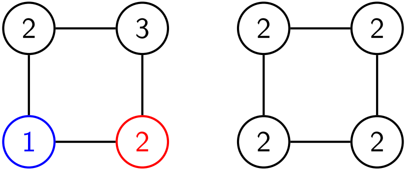
Cyclic Configurations

Example - Continuation



Cyclic Configurations

Example - Continuation



This firing sequences demonstrate how after 4 firings, the initial configuration repeats itself. Therefore, this *Chip-Firing Game* has a *cyclic configuration* and will never reach a *stable state*.

Super-stability of Configurations

Adding or subtracting different numbers of chips from *vertices* disrupts the stability state, making them unstable.

Definition

A condition where configurations in a *Chip-Firing Game* are more stable than other stable configurations.

Super-stability of Configurations

Adding or subtracting different numbers of chips from *vertices* disrupts the stability state, making them unstable.

Definition

A condition where configurations in a *Chip-Firing Game* are more stable than other stable configurations.

When comparing two stable states of the same Chip-Firing Game, if one state requires more chips to be added or subtracted to become unstable, it is considered the more *stable state* or the *super-stable state*.

Infinite-Finite Structure of Chip-Firing Game

Definition

If the game terminates at some point, it has a *finite structure*, and if it still has possible further firings, then it has an *infinite structure*.

Infinite-Finite Structure of Chip-Firing Game

Definition

If the game terminates at some point, it has a *finite structure*, and if it still has possible further firings, then it has an *infinite structure*.

Although there are other techniques, the concept of *stability* is crucial to understanding and analyzing the *Infinite-Finite Structure of Chip-Firing Game*.

Infinite-Finite Structure of Chip-Firing Game

Definition

If the game terminates at some point, it has a *finite structure*, and if it still has possible further firings, then it has an *infinite structure*.

Although there are other techniques, the concept of *stability* is crucial to understanding and analyzing the *Infinite-Finite Structure of Chip-Firing Game*. Thus,

Infinite-Finite Structure of Chip-Firing Game

Definition

If the game terminates at some point, it has a *finite structure*, and if it still has possible further firings, then it has an *infinite structure*.

Although there are other techniques, the concept of *stability* is crucial to understanding and analyzing the *Infinite-Finite Structure of Chip-Firing Game*. Thus,

Definition - Revised

If the game has a *stable state*, then it would have a *finite structure*. However, if it has *unstable states*, then it would have an *infinite structure*.

Infinite-Finite Structure of Chip-Firing Game

However, the previous definition focused on the relative number of *chips* possessed by *vertices* compared to their *degrees*. Instead, now, we will focus on the relative number of *chips* compared to the number of *edges*.

Infinite-Finite Structure of Chip-Firing Game

However, the previous definition focused on the relative number of *chips* possessed by *vertices* compared to their *degrees*. Instead, now, we will focus on the relative number of *chips* compared to the number of *edges*.
1. When the number of chips is less than the number of *edges*, the game always reaches a *stable state*, ensuring *finiteness*.

Infinite-Finite Structure of Chip-Firing Game

However, the previous definition focused on the relative number of *chips* possessed by *vertices* compared to their *degrees*. Instead, now, we will focus on the relative number of *chips* compared to the number of *edges*.
1. When the number of chips is less than the number of *edges*, the game always reaches a *stable state*, ensuring *finiteness*.
2. If the number of chips is at least equal to the number of *edges*, an appropriately chosen initial configuration can lead to an infinite game.

Infinite-Finite Structure of Chip-Firing Game

However, the previous definition focused on the relative number of *chips* possessed by *vertices* compared to their *degrees*. Instead, now, we will focus on the relative number of *chips* compared to the number of *edges*.
1. When the number of chips is less than the number of *edges*, the game always reaches a *stable state*, ensuring *finiteness*.
2. If the number of chips is at least equal to the number of *edges*, an appropriately chosen initial configuration can lead to an infinite game.
3. The game is always infinite when the number of chips exceeds twice the number of *edges* minus the number of *vertices*.

Closing Remark

Thank you so much for listening to me!