The Baker-Campbell-Hausdorff Formula

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Intro

In this talk, we'll look at **Matrix Lie Groups**, which are closed subgroups to the set of invertible $n \times n$ matrices (denoted by $GL_n(\mathbb{C})$). These groups intimately connected to structures called Lie Algebras, which are non-associative algebras with that satisfies the Jacobi identity (which we'll come to later).

Topology on matrices

Let $M_n(\mathbb{C})$ be the set of $n \times n$ matrices.
We define $\|A\|$ for $A \in M_n(\mathbb{C})$ to be the square root of the sum of squares of the modulus of each entry (which are in $\mathbb{C}$). It satisfies the usual properties of a norm (such as the triangle inequality), thought the norm is sub-multiplicative, meaning $\|AB\| \leq \|A\|\|B\|$.
This is the same as the vector norm in $\mathbb{C}^n$, and thus induces a topology on the set of $n \times n$ complex matrices. Hence, we can define topological notions such as connectedness, compactness, or simple-connectedness on sets of matrices.

The Matrix exp and log

Given that

$$e^z = \sum_{i=0}^{\infty} \frac{z^i}{i!}$$

converges for all z, by absolute we can define e^X for a matrix A by just putting A instead of z in the above infinite series.
Similarly, for $\|A - I\| < 1$, we can define

$$\log A = \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(A-1)^i}{i}.$$

It turns out that exp and log are inverses of one another in local neighbourhoods of I and 0 respectively.

Matrix Lie Groups

Let $GL_n(\mathbb{C})$ be the group of $n \times n$ invertible complex matrices. We can define a subspace topology on $GL_n(\mathbb{C})$ by intersecting all open sets of $M_n(\mathbb{C})$ with $GL_n(\mathbb{C})$.
If G is subgroup of $GL_n(\mathbb{C})$ and G is closed with respect to the subspace topology of $GL_n(\mathbb{C})$, then G is a **Matrix Lie Group**.

Examples of Matrix Lie Groups

Trivially, $GL_n(\mathbb{C})$ is Matrix Lie Group. Since $\mathbb{R}$ is a complete subfield of $\mathbb{C}$, $GL_n(\mathbb{R})$ is also a Matrix Lie Group.
The groups $SL_n(\mathbb{C})$ and $SL_n(\mathbb{R})$, that represent the $n \times n$ invertible matrices over $\mathbb{C}$ (or $\mathbb{R}$) with determinant 1, are Matrix Lie Groups.
The set of $n \times n$ orthogonal matrices $A^T A = I$ over $\mathbb{C}$ or $\mathbb{R}$ are further examples. Lastly we have the unitary groups over $\mathbb{C}$; this the group of matrices with $A^H A = I$, where A^H is A^T with all entries complex conjugated.

Matrix Lie Group Homomorphisms

If G and H are two Matrix Lie groups, then if a map $\Phi : G \to H$ is called a **Matrix Lie Group homomorphism** if Φ both a group homomorphism and continuous.
For example, the map $\Phi : GL_2(\mathbb{C}) \to GL_1(\mathbb{C})$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \to ad - bc$$

is a Matrix Lie Group homomorphism. Specifically, this is the determinant.

Lie Algebras

A real vector space $\mathfrak{g}$ is a real **Lie Algebra** if there exists a billinear map $[\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that

- $[X, Y] = [-Y, X]$ (skew-symmetricity)
- $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$ (Jacobi identity)

for all $X, Y, Z \in \mathfrak{g}$.
For example, the cross product over $\mathbb{R}^3$ satisfies the conditions of being a Lie Algebra.
A **Lie Algebra homomorphism** is a linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ between two Lie Algebras such that $[\phi(X), \phi(Y)] = [\phi(X), \phi(Y)]$ for all $X, Y \in \mathfrak{g}$.

The Correspondence between Matrix Lie Groups and Lie Algebras

If G is a Matrix Lie group and

$$\mathfrak{g} = \{X \in M_n(\mathbb{C}) : e^{tX} \in G \text{ for all } t \in \mathbb{R}\},$$

then it can be shown that $\mathfrak{g}$ is a real Lie Algebra with the Lie bracket being the commutator:

$$[X, Y] = XY - YX.$$

We call $\mathfrak{g}$ the Lie algebra *of* G.

The Correspondence between Matrix Lie Group homomorphisms and Lie Algebra homomorphisms

A theorem I prove midway into the paper is that if $\Phi : G \to H$ is a Matrix Lie Group homomorphism, then there exists a unique Lie Algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$ such that

$$e^{\phi(X)} = \Phi(e^X)$$

for all $X \in \mathfrak{g}$.

The Main Result

The converse of the previous result (a Lie Algebra homomorphism giving rise to Matrix Lie Group homomorphism) is in general false, though the main result of my paper is that the converse is true if we assume the domain G is simply-connected.

Theorem

Let G and H are Matrix Lie Groups with Lie Algebras $\mathfrak{g}$ and $\mathfrak{h}$. If G is simply-connected $\phi : \mathfrak{g} \to \mathfrak{h}$ is a Lie Algebra homomorphism, then there exists a unique Lie Group homomorphism $\Phi : G \to H$ such that

$$\Phi(e^X) = e^{\phi(X)}$$

for all $X \in \mathfrak{g}$.

The exp and log functions are inverses from G to $\mathfrak{g}$

By the definition of the Lie Algebra $\mathfrak{g}$, $e^X \in G$ for all $X \in \mathfrak{g}$, which means $\exp : \mathfrak{g} \to G$ makes sense.
It turns out that exp and log are inverses of one another locally on neighbourhoods of I and 0 in G and $\mathfrak{g}$ respectively.

Theorem

For $0 < \epsilon < \log 2$, let $U_\epsilon = \{X \in M_n(\mathbb{C}) | \|X\| < \epsilon\}$, and let $V_\epsilon = \exp(U_\epsilon)$. Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$. Then there exists $\epsilon \in (0, \log 2)$ such that for any $A \in V_\epsilon$, we have $A \in G$ if and only if $\log A \in \mathfrak{g}$.

Baker-Campbell-Hausdorff Formula

The function

$$g(z) = \frac{\log z}{1 - \frac{1}{z}}$$

is analytic (meaning it has a power series) for complex z such that $|z - 1| < 1$. By absolute convergence, we can extend this the power series of the above function g to complex matrices A with $\|A - I\| < 1$.

For the BCH formula, we would actually need to input a (linear) operator that takes matrices to matrices (or elements of $M_n(\mathbb{C})$). Such operators can be understood as an element of $M_{n^2}(\mathbb{C})$ by viewing elements of $M_n(\mathbb{C})$ itself as vectors in $\mathbb{C}^{n^2}$.

Baker-Campbell-Hausdorrf Formula

The most important ingredient that is used in the proof is the **Baker-Campbell Hausdorff** formula, which states that for $n \times n$ complex matrices X and Y with $\|X\|$ and $\|Y\|$ sufficiently small,

$$\log(e^X e^Y) = X + \int_{t=0}^{1} g(e^{\mathrm{ad}_X} e^{t\,\mathrm{ad}_Y})dt,$$

where ad_X is the map that sens Y to $[X, Y] = XY - YX$.
A series from of the above integral formula would be:

$$\log(e^X e^Y) = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] + \cdots$$

Note that this is an infinite series of nested brackets.

Local homomorphisms

Using the BCH formula, one can prove that a *local homomorphism* is possible, as in a $\Phi' : U \to H$ exists with $U \subseteq G$ being a small neighbourhood of I, and

$$\Phi'(e^X) = e^{\phi(X)},$$

for all $X \in \mathfrak{g}$, along with $\Phi(A)\Phi(B) = \Phi(AB)$ for all $A, B \in U$ such that $AB \in U$. Note that G needn't be simply-connected for the existence of such a local homomorphism.

Using the simply-connected nature of G, one can extend such a local homomorphism to a global homomorphism.

Proof of the existence for local homomorphism

Firstly, observe that Φ' must be continuous as it is the composition of the continuous function log, ϕ (which is continuous as it's a linear map), and exp.

Let $\epsilon > 0$ be small enough such that the diffeomorphism between exp and log over G and $\mathfrak{g}$ holds, and small enough so that for any $A, B \in U_\epsilon$, the BCH formula holds for $X = \log A$ and $Y = \log B$, as well as $\phi(X))$ along with $\phi(Y)$ (note that $X, Y \in \mathfrak{g}$ and $\phi(X), \phi(Y) \in \mathfrak{h}$) . If $AB \in U_\epsilon$, then
If $AB \in U_\epsilon$, then

$$\Phi'(AB) = \Phi'(e^X e^Y) = e^{\phi(\log(e^X e^Y))}.$$

Proof of the existence for local homomorphism

We now apply the BCH formula, which can be expressed as an infinite series of nested brackets. We have

$$\begin{aligned}
&\phi(\log(e^X e^Y)) \\
&= \phi(X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] + \cdots) \\
&= \phi(X) + \phi(Y) + \frac{1}{2}[\phi(X), \phi(Y)] + \\
&\frac{1}{12}[\phi(X), [\phi(X), \phi(Y)]] - \frac{1}{12}[\phi(Y), [\phi(X), \phi(Y)]] + \cdots \\
&= \log(e^{\phi(X)} e^{\phi(Y)}),
\end{aligned}$$

where we've used the fact that $\phi([X, Y]) = [\phi(X), \phi(Y)]$. We now get

$$\Phi'(AB) = e^{\phi(\log(e^X e^Y))} = e^{\log(e^{\phi(X)} e^{\phi(Y)})} = e^{\phi(X)} e^{\phi(Y)} = \Phi'(A)\Phi'(B),$$

which proves the result. Note that $\exp \circ \log$ is the identity in this case as X and Y are sufficiently close to I.

References

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