Presentation on (Stone) Weierstrass theorem

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History of Weierstrass theorem

- There are two versions!
  - Approximation theorem in the real space (proved in 1937 by Karl Theodore Wilhelm Weierstrass).
  - Simplified and extended to Stone-Weierstrass theorem by Marshall Harvey Stone in 1948.
- Proofs are different since they focus on different aspects of math, but the Stone theorem relies on the approximation theorem.

Figure: Weierstrass

Figure: Stone

Statement of the Weierstrass theorem

Suppose f is a continuous complex-valued function defined on the real interval $[a, b]$, there is a sequence of polynomials $p_n(x)$ that converges uniformly to $f(x)$ on $[a, b]$.

- Powerful since it can approximate any functions once continuous.
- Better than Taylor Series in getting an error smaller than arbitrary number.

Terms in the theorem explained

- Converges uniformly: in a sequence f_n of functions, for every $\epsilon > 0$, there is a positive integer N such that for every integer $n \geq N$, we have $|f_n(x) - f(x)| < \epsilon$
- Continuous (uniformly since on a closed interval): for a function f, to ensure $|f(a) - f(b)| < \epsilon$ for a $\epsilon > 0$, we only need to ensure $|a - b| < \delta$ for a $\delta > 0$.

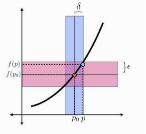

Figure: Uniformly continuous

Differences between two versions

- The closed interval $[a, b]$ is replaced with a compact space in the Stone theorem.
- The continuous function is replaced with an algebra of real, bounded continuous functions.

Ideas to prove the approximation theorem

For each f and a $\epsilon > 0$, we try to find a polynomial $p(x)$ that is really close to f at any point in the interval. Since we have the interval, there is a bound on the value of f, so we could utilize that to get a bound. Besides, we could "sample" points of the function by using Bernstein polynomial. It is shown later that $B_n(x, f)$ is a approximation that works.

Bernstein polynomial: key in the proof

Bernstein polynomial is defined as:

$$B_n(x, f) = \sum_{k=0}^{n} f(\frac{k}{n}) \binom{n}{k} x^k (1-x)^{n-k}$$

There are several properties that make it desirable:

1. We have when $|f(x)| \leq |g(x)|$ for every x, we have $B_n(x, f) \leq B_n(x, g)$
2. $B_n(x, 1)$, where 1 is the constant function, is equal to 1. This property helps the next one.
3. $B_n(x, f - a) = B_n(x, f) - a$, where a is a constant.
4. $B_n(x, (x-e)^2) = (1 - \frac{1}{n})x^2 + \frac{1}{n}x - 2ex + e^2$

Proof of properties for Bernstein polynomial

1. The first property 1 is because every term in the Bernstein polynomial for f is smaller.
2. The second property is because of binomial theorem.
3. Third one $B_n(x, f-a) = \sum_{k=0}^{n}(f-a)(\frac{k}{n})\binom{n}{k}x^k(1-x)x^{n-k}$, separate the constant and use binomial series.
4. Last property on the next page.

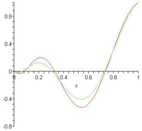

Figure: Bernstein polynomial

Properties 4 proof for the Bernstein polynomial

Key idea: manipulation of separate binomial terms:

$$\begin{aligned}
B_n(x,(x-e)^2) &= \sum_{k=0}^{n}(\frac{k}{n}-e)^2\binom{n}{k}x^k(1-x)^{n-k} \\
&= \sum_{k=0}^{n}(\frac{k^2}{n^2}-\frac{2k}{n}e+e^2)\binom{n}{k}x^k(1-x)^{n-k} \\
&= x^2\sum_{k=2}^{n}\frac{n-1}{n}\binom{n-2}{k-2}x^{k-2}(1-x)^{n-k} \\
&+ x\sum_{k=1}^{n}(\frac{1}{n}-2e)\binom{n-1}{k-1}x^{k-1}(1-x)^{n-k}+e^2 \\
&= (1-\frac{1}{n})x^2+\frac{1}{n}x-2ex+e^2
\end{aligned}$$

Proof page 1: setup

Setup: constrain the interval to be $[0, 1]$, since we can scale back the results later.
Use the definition of uniformly continuous (suppose we fix x):

$$|x - y| \leq \delta \implies |f(x) - f(y)| \leq \frac{\epsilon}{2} \to \exists e, |f(x) - f(e)| \leq \frac{\epsilon}{2}$$

and define the norm of the function:

$$M = \|f\|_{\infty} = \max_{x \in [0,1]} |f(x)|$$

Proof page 2: create unconditional bounds

For $|x-e| \geq \delta$ ($|f(x)-f(e)|$ might be bigger than ϵ), we have

$$|f(x)-f(e)| \leq |f(x)|+|f(e)| \leq M+M \leq 2M \leq 2M(\frac{x-e}{\delta})^2+\frac{\epsilon}{2}$$

Therefore, in any case:

$$|f(x)-f(e)| \leq 2M(\frac{x-e}{\delta})^2+\frac{\epsilon}{2}$$

it helps to constrain the difference between f(x) and f(e) no matter what conditions they have.

Proof page 3: use the properties

Use the difference property and how value of the polynomial increases when the function inside increases, we have

$$
\begin{aligned}
&|(B_n(x, f) - f(e))| \\
&= |B_n(x, f - f(e))| \leq B_n(x, 2M(\frac{x - \epsilon}{\delta})^2 + \frac{\epsilon}{2}) \\
&= \frac{2M}{\delta^2} B_n(x, (x - e)^2) + \frac{\epsilon}{2}
\end{aligned}
$$

Use the property about the Bernstein polynomial when f is quadratic, we have

$$B_n(x, (x - e)^2) = x^2 + \frac{1}{n}(x - x^2) - 2ex + e^2$$

Putting it all together

After the algebra manipulation, we have

$$|B_n(x,f) - f(e)| \leq \frac{\epsilon}{2} + \frac{2M}{\delta^2}\frac{1}{n}(e - e^2) \leq \frac{\epsilon}{2} + \frac{M}{2\delta^2 n}$$

Notice that in the right hand side, we can change the value of n. Choose n that is big enough finishes the problem.

Beyond

- Stone-Weierstrass theorem's proof is shorter, but includes work around topology structures.
- Stone-Weierstrass theorem has many applications. Either on other algebraic structures of series.
- Read more from my paper.

Acknowledgement

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