Calculus of Variations and Further

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Euler Circle

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Root: a physics question

In June of 1696, John (also known as Johann or Jean) Bernoulli challenged the greatest mathematicians of the world to solve the following new problem (Bernoulli, 1696; Goldstine, 1980):

> *Given points A and B in a vertical plane to find the path AMB down which a movable point M must, by virtue of its weight, proceed from A to B in the shortest possible time[Kot14, Chapter 1].*

First steps of solving

In the question, we have:

1. $\frac{1}{2}mv^2 - mgy = 0$
2. $dt = \frac{dl}{v}$
3. $dl = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + (\frac{dy}{dx})^2}\ dx$

So, we could get $v = \sqrt{2gy}$.

Combine all above equations together, we have: $dt = \frac{\sqrt{1+(y')^2}}{\sqrt{2gy}}\ dx$

$T = \int dt = \frac{1}{\sqrt{2g}} \int_0^{x_2} \frac{\sqrt{1+(y')^2}}{\sqrt{y}}\ dx$

Here, we aim to get the minimum value of T.

This lead to our first definition:

Definition

A functional is an operator that maps functions to real numbers[Kot14, Chapter 2].

As T is a functional, to solve the problem, calculus of variations is introduced.

In this section, I'm going to explore the standard forms of calculus of variations and solve the simplified problem of the curve of fastest descent. Remember, we have $T = \frac{1}{\sqrt{2g}} \int_0^{x_2} \frac{\sqrt{1+(y')^2}}{\sqrt{y}}\ dx$. To simplify the problem further (so that we can focus on the maths idea), we can ignore the constant coefficient $\frac{1}{\sqrt{2g}}$, and we can define another functional as:

$$I = \int_0^{x_2} \frac{\sqrt{1+(y')^2}}{\sqrt{y}}\ dx.$$

Introduce General Function

Also, to make the writing simpler, here we can write $\frac{\sqrt{1+(y')^2}}{\sqrt{y}}$ as $f(y, y'; x)$ since it is a functional with variations y and y' and is also indirectly relevant to x.
Now, let's start with a more general form of functional. Suppose we have:

$$I = \int_{x_1}^{x_2} f(y, y'; x)\ dx.$$

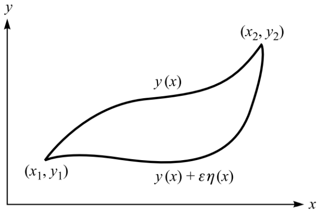
Suppose we are calculating the minimum.

Suppose the function y for minimum I is $y(0, x)$, where $\epsilon = 0$ and x is the independent variable. To vary y by varying ϵ, we can define y as $y(\epsilon, x) = y(0, x) + \epsilon\eta(x)$, where $\eta(x)$ is an arbitrary continuous function about independent variable x.

For any none-zero ϵ, I would be larger.

We have to satisfy the boundary conditions, $y(\epsilon, x_1) = y(0, x_1)$ and $y(\epsilon, x_2) = y(0, x_2)$, as the distortion of function y should not affect the initial position A and final position B in this context (recall the description at the beginning of the essay!), and similar boundary conditions should be recognized in other problems of calculus of variations. So, we have boundary conditions:

$$\eta(x_1) = 0 \text{ and } \eta(x_2) = 0.$$



Here we can see I is now only dependent with a single parameter ϵ.
To make the minimum I, as in normal case of functions, we should have extreme when $\epsilon = 0$. Thus, the equation should hold:

$$\frac{\partial I}{\partial \epsilon} \mid_{\epsilon=0} = 0.$$

Simplify the equation. As in a classic mechanic system, the partial differentials must be continuous, which makes the order of differential and integral matter little, we have:

$$\frac{\partial I}{\partial \epsilon} = \frac{\partial}{\partial \epsilon} \int_{x_1}^{x_2} f(y(\epsilon, x), y'(\epsilon, x); x) \; dx = \int_{x_1}^{x_2} \frac{\partial}{\partial \epsilon} f(y(\epsilon, x), y'(\epsilon, x); x) \; dx.$$

Then, we can apply chain rule in the partial differential, and we get:

$$\frac{\partial I}{\partial \epsilon} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \epsilon} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \epsilon}\right) dx.$$

We have:

$$\frac{\partial y}{\partial \epsilon} = \eta(x)$$

$$\frac{\partial y'}{\partial \epsilon} = \frac{\partial}{\partial \epsilon}(y'(0,x) + \epsilon\eta'(x)) = \eta'(x)$$

To cancel out $\eta'(x)$, we can expand $\int_{x_1}^{x_2} \frac{\partial f}{\partial y'}\eta'(x)\ dx$ using integral by parts. Thus, let $u = \frac{\partial f}{\partial y'}$ and $dv = \eta'(x)\ dx$, and we can deduce:

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y'}\eta'(x)\ dx = \int_{x_1}^{x_2} u\ dv = [uv]_{x_1}^{x_2} - \int_{x_1}^{x_2} v\ du = [\frac{\partial f}{\partial y'}\eta(x)]_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}(\frac{\partial f}{\partial y'})\eta(x)\ dx.$$

Recall the boundary conditions. We have:

$$[\frac{\partial f}{\partial y'}\eta(x)]_{x_1}^{x_2} = \eta(x_1)\frac{\partial f}{\partial y'}\ |_{x_1} - \eta(x_2)\frac{\partial f}{\partial y'}\ |_{x_2} = 0.$$

Then, we have:

$$\frac{\partial I}{\partial \epsilon} = \int_{x_1}^{x_2} \frac{\partial f}{\partial y}\eta(x)\ dx - \int_{x_1}^{x_2} \frac{d}{dx}(\frac{\partial f}{\partial y'})\eta(x)\ dx = \int_{x_1}^{x_2}[\frac{\partial f}{\partial y} - \frac{d}{dx}(\frac{\partial f}{\partial y'})]\eta(x)\ dx.$$

Since $\eta(x)$ is an arbitrary function, when $\frac{\partial I}{\partial \epsilon} = 0$, we have:

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0.$$

And it is the so-called Euler-Lagrange equation. Note that it is only a necessary condition for I to be an extreme value. Thus, the solution of the Euler-Lagrange equation may not yield the minimizing curve. Ordinarily we must verify whether or not this solution yields the curve that actually minimizes the integral, but frequently physical or geometrical considerations enable us to tell whether the curve so obtained makes the integral a minimum or a maximum[CHO03, Chapter 8].

Recall that:

$$I = \int_0^{x_2} \frac{\sqrt{1+(y')^2}}{\sqrt{y}}\, dx$$
$$f(y, y'; x) = \frac{\sqrt{1+(y')^2}}{\sqrt{y}}.$$

When the Euler-Lagrange equation holds, we have $\frac{d}{dx}(\frac{\partial f}{\partial y'}) = \frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial y'} = \frac{\partial}{\partial y'} \frac{\sqrt{1+(y')^2}}{\sqrt{y}} = \frac{1}{\sqrt{y}} \frac{y'}{\sqrt{1+(y')^2}},$$
$$\frac{d}{dx}(\frac{\partial f}{\partial y'}) = \frac{-\frac{1}{2}(y')^2(1+(y')^2)+yy''}{y^{\frac{3}{2}}(1+(y')^2)^{\frac{3}{2}}},$$
$$\frac{\partial f}{\partial y} = -\frac{1}{2}\sqrt{1+(y')^2}\; y^{-\frac{3}{2}},$$

$$\text{LHS} = \int \frac{2y''y'}{1+(y')^2} = \int \frac{1}{1+(y')^2}\, d(1+(y')^2) = \ln(1+(y')^2);$$
$$\text{RHS} = -\int \frac{1}{y} y' = -\ln y + \ln k.$$

So,

$$(y')^2 = \frac{k}{y} - 1.$$
$$y' = \frac{dy}{dx} = \pm\sqrt{\frac{k}{y} - 1}$$

Thus, we can get:

$$\sqrt{\frac{y}{k-y}}\, dy = \pm dx$$

Let $y = \frac{k}{2}(1 - \cos u)$, so $dy = \frac{k}{2}\sin u\ du$. We have:

$$\sqrt{\frac{y}{k-y}}\ dy = \frac{k}{2}(1 - \cos u)\ du$$

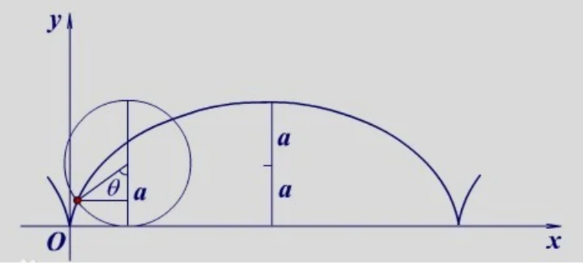
Thus,

$$\frac{k}{2}(1 - \cos u)\ du = \pm dx$$
$$\int \frac{k}{2}(1 - \cos u)\ du = \pm \int dx$$
$$x = \pm\frac{k}{2}(u - \sin u) + const.$$

Recall the boundary condition that when $y = 0$ i.e., $u = 0$, $x = 0$, so $const = 0$.

The Parameter Equations of the Curve of Fastest Descent

$$\begin{cases} x = a(\theta - \sin\theta) \\ y = a(1 - \cos\theta) \end{cases}$$



In the geometric optics, we have the well known theorem of Format Principle.

Theorem

The optical length of the path followed by light between two fixed different points is the global minima. The optical length is the physical length multiplied by the refractive index of the medium.

In this way, the minimum of time is converted to the minimum of $\frac{1}{c}\int_A^B n\, ds$, which can be defined as optical path length.

Definition

The optical path length S of a ray from A to B is defined following integral $S = \frac{1}{c}\int_A^B n\, ds$, related to the travel time T by $S = cT$ [RGGA20, Chapter 2.5].

Law of Reflection:

$\sin\theta_1 = \sin\theta_2$, i.e., $\theta_1 = \theta_2$

Law of Refraction:

$n_1 \sin\theta_1 = n_2 \sin\theta_2$

Let the movement of the light ray $q = q(\tau)$ in the $(\tau, q) - plane$. Let $n(q)$ be the index of refraction at the point q[EZ04].

If we introduce $L(q, q', \tau) = \frac{n(q)}{c}\sqrt{1 + (q')^2}$, we can apply the conclusion of calculus of variations to get the condition needed to satisfy Fermat's principle:

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial q'}\right) - \frac{\partial L}{\partial q} = 0.$$

Thus we have the Euler-Lagrange equation in the optical case as:

$$\frac{d}{d\tau}\frac{nq'}{\sqrt{1+(q')^2}} = \sqrt{1+(q')^2}\frac{\partial n}{\partial q}.$$

To simplify notations we choose the units of measurement so that $c = 1$.

We can also intrduce the Hamiltonian canonical equations into optics. Also, thus methods can be applied to solve problems like the optical fibers. All these are discussed in my paper! And I hope you can enjoy it.

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