Nonstandard Methods and Applications in Ramsey Theory

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July 11, 2022

A Brief History

- Gottfried Leibniz uses infinitesimals in his development of calculus (1700)
- Infinitesimal approach can't be rigorously defined, criticized
- Abraham Robinson revives it and gives it a rigorous treatment (1960)
- Nonstandard analysis can be now applied to many other areas of mathematics.

Figure: Leibniz

What is Nonstandard Analysis?

Nonstandard analysis has two central concepts:

1. Every mathematical object X has some corresponding ${}^*X$ (labled the **nonstandard-extension**).
2. ${}^*X$ shares the same elementary properties as X.

We call this the **transfer principle,** and the relation between X and ${}^*X$ is referred to as the **star map**.

Transfer Property

We can formalize the transfer property as follows.

Theorem (Transfer Property)

Let $P(A_1, \ldots, A_n)$ be some elementary property of the mathematical objects $A_1, \ldots, A_n$. Then, we have

$$P(A_1, \ldots, A_n) \Longleftrightarrow P(A_1^*, \ldots, A_n^*).$$

Elementary Properties

Elementary properties of X deal with the elements of X, like

- associativity and commutativity in $\mathbb{R}$.

Non-elementary properties deal with higher level structures, such as:

- the Well-Ordering Principle of $\mathbb{Z}$.

Hyperreals

The hyperreals $^*\mathbb{R}$ are a number system which contains **infinitesimal numbers** ϵ such that for all $n \in \mathbb{R}$:

$$|\epsilon| < \frac{1}{n},$$

and **infinitely large** numbers Ω such that

$$|\Omega| > n.$$

They preserve the elementary properties of the real numbers.

Construction of the Hyperreals

Definition

A *filter* over some nonempty set I is a nonempty collection $\mathcal{F} \subseteq \mathcal{P}(I)$ such that:

- if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$;
- if $A \in \mathcal{F}$ and $A \subseteq B \subseteq I$, then $B \in \mathcal{F}$.

Definition

An *ultrafilter* $\mathcal{U}$ over some I is a filter such that for every $A \subseteq I$, either A or A^c is a member of $\mathcal{U}$. A *nonprincipal ultrafilter* contains no finite sets.

Construction of the Hyperreals Part II

Let $\mathbb{R}^N$ denote the set of all sequences of real numbers, and let $\mathcal{U}$ be a nonprincipal ultrafilter.

Definition

Two sequences $r, s \in \mathbb{R}^N$ are equivalent if and only if their elements are equivalent at a large number of places,

$$\{r_n = s_n \mid n \in \mathbb{N}\} \in \mathcal{U}.$$

Definition

The *equivalence class* of r consists of all sequences equal to r. It is denoted by $[r]$.

Construction of the Hyperreals Part III

The hyperreals ${}^*\mathbb{R}$ are the set of distinct equivalence classes of $\mathbb{R}^N$, that is

$$^*\mathbb{R} = \{[r] \,|\, r \in \mathbb{R}^N\}.$$

The hyperreals are an ordered field satisfying the field axioms, and are the nonstandard extension of the real numbers. Some real number n corresponds to the sequence $\langle n, n, \ldots \rangle \in {}^*\mathbb{R}$.

The hyperintegers ${}^*\mathbb{Z}$ are a subset of ${}^*\mathbb{R}$ consisting of the integer corresponding hyperreals. The hypernaturals ${}^*\mathbb{N}$ are the positive hyperintegers.

Standard Parts of Hyperreals

Every finite $\xi \in {}^*\mathbb{R}$ is arbitrarily close to some real number n such that we define

$$\operatorname{st}(\xi) = n.$$

Let $\xi_1, \xi_2 \in {}^*\mathbb{R}$.

- $\operatorname{st}(\xi_1 + \xi_2) = \operatorname{st}(\xi_1) + \operatorname{st}(\xi_2)$.
- $\operatorname{st}(\xi_1 \xi_2) = \operatorname{st}(\xi_1) \operatorname{st}(\xi_2)$.

The Nonstandard Derivative

Definition

Let dx be an infinitesimal hyperreal. Then, the derivative of the function $f(x)$ is given by

$$f'(x) = \text{st}\left(\frac{f(x+dx) - f(x)}{dx}\right).$$

This is quite similar to the traditional definition, the main difference being the lack of limits.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

The Infinitude of Primes ft. Nonstandard Methods

Let Π denote the set of all prime numbers. The enlargement $^*\Pi$ is its nonstandard extension. This has nonstandard members, and we can use this to derive a contradiction (if Π is finite, $^*\Pi = \Pi$.).

Let N be a hypernatural number that is divisible by every member of $\mathbb{N}$ and let q be a member of $^*\Pi$ that divides $N + 1$. We notice that q cannot be a member of Π, as by our assumption it would divide N and the number $N + 1 - N = 1$, which is false for any prime. Therefore, q is nonstandard, and therefore Π is infinite.

An Introduction to Ramsey Theory

- Named after British mathematician Frank P. Ramsey (1903 - 1930)
- Focused on finding "order" in arbitrary structures: how big does something have to be for a property to hold?

Figure: Frank Plumpton Ramsey

Ramsey's Theorem

Theorem (Infinite Ramsey's Theorem)

Let X be some infinite set and $X^{[m]}$ be the set of $m-$sized subsets of X. For some arbitrary $c \in \mathbb{N}$, for any arbitrary coloring $C_1 \cup \cdots \cup C_c$ of $X^{[m]}$ there exists some infinite $A \subseteq X$ such that $A \subseteq C_i$, for some i.

This is most well known in the context of graphs: for any infinite graph G and an arbitrary number of finite edge colorings of the graph, there exists a connected monochromatic infinite graph in G.

Proof (Outline)

- Choose some infinite v such that $\{v, {}^*v\} \in {}^{**}C$.
- By transfer we can pick some q_1 such that $\{q_1, v\} \in {}^*C$.
- From the previous, we can pick some $q_2 > q_1$ such that $\{q_2, v\} \in {}^*C$ and $\{q_1, q_2\} \in C$.
- We can proceed to arbitrarily pick some q_n such that $\{q_1, q_n\}, \ldots, \{q_{n-1}, q_n\} \in C$, which creates our fully connected infinite monochromatic graph.

Hindman's Theorem

Theorem (Hindman's Theorem)

For every finite coloring of $\mathbb{N}$ there exists an infinite $X = (x_1, \ldots, x_n)$ such that all finite sums $FS(X) = \{x_F = \sum_{i \in F} x_i \mid F \subset \mathbb{N} \text{ finite}\}$ are monochromatic.

"Anyone with a very masochistic bent is invited to wade through the original combinatorial proof." (Neil Hindman)

Ultrafilters Revisited

Definition

Two hypernaturals ξ, ζ are "u-equivalent" (represented by $\sim$) if they generate the same ultrafilter. An ultrafilter generated by a hypernatural number ξ is represented by

$$\mathcal{U}_\xi = \{A \subseteq \mathbb{N} \mid \xi \in {}^*A\}.$$

Definition

We define the pseudo-sum $\oplus$ operation on ultrafilters generated by hypernaturals as such:

$$A \in \mathcal{U} \oplus \mathcal{V} \iff \{n \mid A - n \in \mathcal{V}\} \in \mathcal{U}.$$

Ultrafilters Continued

Definition

An idempotent ultrafilter $\mathcal{U}$ is idempotent if

$$\mathcal{U} \bigoplus \mathcal{U} = \mathcal{U}.$$

Note that because ultrafilters can be generated by hypernaturals, an idempotent hypernatural simply generates an idempotent ultrafilter.

Outline of Proof

- Pick an idempotent $v \in {}^*\mathbb{N}$ and let C be the color with $v \in {}^*C$.
- Pick $x_1 \in C$ such that $x_1 + v \in {}^*C$.
- Inductively, assume that we defined $X_1 < \ldots < x_n$ such that $x_F = \sum_{i \in F} x_i \in C$ and $x_F + v \in {}^*C$ for every $F \subseteq \{1, \ldots, n\}$.
- Since $v \sim (v + {}^*v)$, (by idempotent properties) we also have $x_F + v + {}^*v \in {}^{**}C$.
- Since $x_F + v \in {}^*C$ and $x_F + v + {}^*v \in {}^{**}A$, by the transfer property we find that $x_{n+1} > x_n$ such that $x_F + x_{n+1} \in C$ and $x_F + x_{n+1} + v \in {}^*C$ for every F.

Partition Regularity of Diophantine Equations

Definition

An equation $f(X_1, \ldots, X_n) = 0$ is partition regular (PR) on $\mathbb{N}$ if for every finite coloring of $\mathbb{N}$ there exist a monochromatic solution, i.e. monochromatic elements $x_1, \ldots, x_n$ such that $F(x_1, \ldots, x_n) = 0$.

There are several prominent theorems in this area, including:

- **Schur's Theorem**: In every finite coloring of $\mathbb{N}$ one finds monochromatic triples $a, b, a+b$.
  From this, $\mathrm{X} + \mathrm{Y} = \mathrm{Z}$ is PR.
- **van der Waerden's Theorem**: In every finite coloring of $\mathbb{N}$ one finds arbitrarily long arithmetic progressions.

Nonstandard characterization of PR

Theorem (Nonstandard characterization)

An equation $f(X_1, \ldots, X_n) = 0$ *is partition regular on* $\mathbb{N}$ *if there exist* $\xi_1 \sim \cdots \sim \xi_n$ *in* ${}^*\mathbb{N}$ *such that* ${}^*f(\xi_1, \ldots, \xi_n) = 0$.

Bergelson-Hindman

Theorem (Bergelson-Hindman)

Let $\mathcal{U}$ be an idempotent ultrafilter. Then every $A \in 2\mathcal{U} \oplus \mathcal{U}$ contains an arithmetic progression of length 3. Therefore, $X - 2Y + Z = 0$ is partition regular.

Note that $v \sim v + {}^*v$ for idempotents.

Proof Outline Again

Let $v \in {}^*\mathbb{N}$ be such that $\mathcal{U}$ is the ultrafilter generated by v. Therefore, $v \sim v + {}^*v$. So, let

- $\xi = 2v + {}^{**}v$.
- $\zeta = 2v + {}^*v + {}^{**}v$.
- $\mu = 2v + 2{}^*v + {}^{**}v$.

These are u-equivalent numbers of ${}^{***}\mathbb{N}$ that generate $\mathcal{V} = 2\mathcal{U} \oplus \mathcal{U}$. For every $A \in \mathcal{U}$, the elements $\xi, \zeta, \mu \in {}^{***}A$ form a 3-term arithmetic progression, so by transfer there exists a 3-term arithmetic progression in A.

Acknowledgements

Thank you to Maxim for being an awesome TA and helping me get through the more difficult topics!, and thank you Simon for giving me the opportunity to learn about a subject so deeply!