CIRCUMSCRIBING CHARACTERISTICS OF JORDAN CURVE

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Abstract

This paper aims to teach readers the mathematical definition of simple closed curve as well as specific properties of it. All the proofs in this paper are about the ability of a Jordan curve to circumscribe different figures such as square, rectangle, and rhombus sometimes under certain conditions like symmetry. This paper assumes basic knowledge in geometry like symmetry, similarity and congruence of figures, rotations, and some trivial notations in calculus.

Contents

1. Introduction

This paper includes descriptions and proofs of simple closed curves; simple closed curve will be referred often as Jordan curve. Jordan curves are curves that can be drawn without lifting the pencil up on a paper and begins and ends at the same point by non-rigorous definition as more rigorous definition will be provided in later sections. My interests in this specific curve came from seeing various attempts to prove that a Jordan curve always inscribes a square on the internet. In this paper, I intend to introduce and explain proofs that provide various characteristics of Jordan curve such as this: a Jordan curve inscribes many triangles similar to an arbitrary triangle. I hope you learn a lot more about Jordan curves and be more interested in this topic through this paper! Also, all the sections before

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fundamentals of topology require shallow understandings in geometry. But, the chapters including fundamentals of topology and later are more advanced topics that require more mathematical intuition. Although, I hope that later sections can be understood with careful reading and thinking. Note: we will generally focus on the plane $\mathbb{R}^2$ in this paper, and the paper will specifically mention that another plane being referred to if that is the case; all the Jordan curves that are mentioned in this paper in the proofs are "nice enough" Jordan curves as theorem 2.3 states in preliminaries.

Here are some of the most important theorems that I will prove in the paper.

Theorem 1.1. *$\forall$Jordan curve J, symmetric about a point inside, $\exists$ four points P_1, P_2, P_3, P_4 on J such that quadrilateral $P_1P_2P_3P_4$ is a square*

Theorem 1.2. *$\forall$ Jordan curve J in $\mathbb{R}^3$ with point $p \in J$ such that $J'(p) = k$ where $k \in \mathbb{R}$, $\exists$ points $P_1, P_2, P_3 \in J$ such that $\triangle P_1P_2P_3 \sim T$ where T is an arbitrary triangle.*

Theorem 1.3. *For Jordan curve J, $\exists$ four points $P_1, P_2, P_3, P_4 \in J$ such that quadrilateral $P_1P_2P_3P_4$ is a rectangle*

2. Preliminaries

Definition 2.1. Simple closed curve is the image of f([0,1]) where f:[0,1] $\to \mathbb{R}^n$ one-to-one, but f(0) = f(1)

Definition 2.2. A Jordan curve J is"nice enough" if $\forall$ ordered pairs p $\in$ J, $\exists$ a coordinate system such that one possible set X $\in$ J with p $\in$ X has all elements $\gamma \in$ domain of X $\to \delta$ $\in$ co-domain of X.

Theorem 2.3. *Stromquist's theorem states that only if Jordan curve J is "nice enough", a general polygon $P_1P_2P_3 \dots P_n$ is on J, meaning $P_1, P_2, P_3, \dots, P_n \in J$.*

The theorem above is very important because this shows that all the Jordan curves that have potential of circumscribing a figure is "nice enough".

Examples and non-examples of Jordan curve

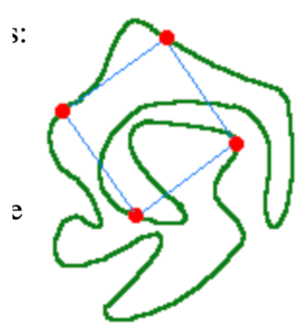


Figure 1. An example of simple closed curve

As you can see above, the curve is an example of simple closed curve that is "nice enough". You can imagine coordinate axes x and y, y being orthogonal to x in the plane $\mathbb{R}^2$ such that

Figure 2. An example of a point in J being a part of function

every point on this curve is a part of continuous function. Below is an example of how you can draw coordinate axes to make a part of figure 1 be a continuous function.

It is basically safe to assume that every curve that you draw on a piece of paper that begins and ends at the same point is "nice enough". However, there are some extremely rare cases of a Jordan curve being not "nice enough". Often times, not "nice enough" curves are curves that are VERY fractal like. Below is an example of how a not "nice enough" Jordan curve might look like.

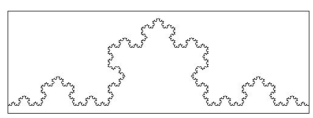


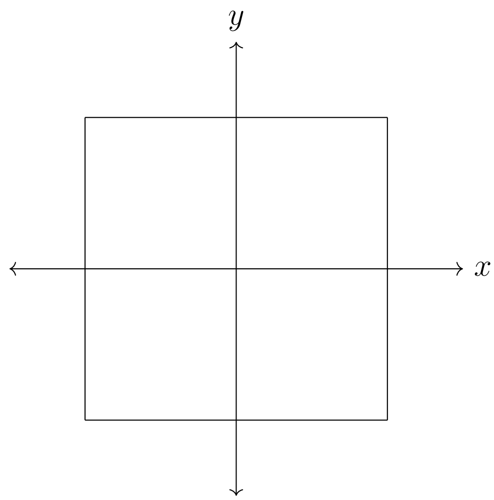
Figure 3. An example of fractal-like curve

As you can see in examples of extreme fractal-like curve, it is hard to find a coordinate system can be set up so that the part of function containing that point is defined as a function. Thus, an extreme version of figure 3 is an example of not "nice enough" Jordan curve.

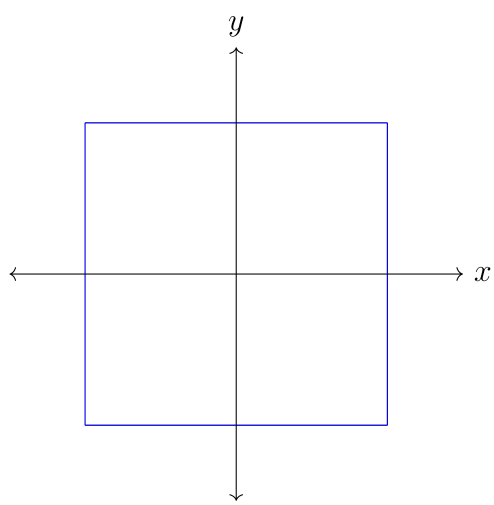
3. Symmetric Jordan curve circumscribing a square

Theorem 3.1. *$\forall$Jordan curve J, symmetric about a point inside, $\exists$ four points P_1, P_2, P_3, P_4 on J such that quadrilateral $P_1P_2P_3P_4$ is a square*

We first need to build intuition for this proof and we will do that by using a simple example of Jordan curve that is symmetric about a point inside of it, a square.



You can easily see that this square is symmetric about origin. Since a function is defined at every point on this square, we can claim that this square is a Jordan curve that is "nice enough", thus we can use this to prove theorem 2.3. And, let's call this square J. To prove that another square is inscribed in this square, let's first rotate the square $pi/2$ radians in counter clockwise direction. Let that function be defined as f:$\mathbb{R}^2 \to \mathbb{R}^2$ that rotates every p $\in$ domain $\mathbb{R}^2$ is rotated $pi/2$ radians in counter clockwise direction. J and f(J) on the same coordinate axes are drawn below.



You see that the J and f(J) exactly overlap one another. As an example, (1,2) is on both J and f(J), and we know that (2,-1) is on J because (1,2) is mere right angle counter clockwise rotation of (2,-1). Then, we know (-2,1) and (-1,-2) are also on J using odd function property. We now can prove that J contains a square because the four points form four lines that are perpendicular to each other if they are adjacent and are same length. An interesting thing to note is that since f(J) and J are identical, we can see that the step of proving a Jordan curve contains an inscribed square can be done for every point on J, so a square has infinitely many inscribed squares. The only extension of general proof of any Jordan curve having an inscribed square is that we need to prove that J and f(J) meet at least one point rather than assuming they meet at every point like we did with the square example. After that same process as done in the square example can be done to the generalized Jordan curve to prove there exists an inscribed square in J.

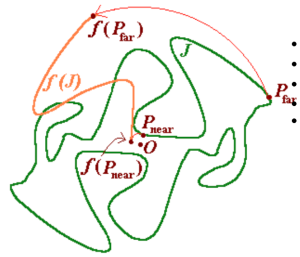


Figure 4. Possibilities of an intersection of J and f(J)

Proof. Let f be one to one function: $\mathbb{R}^2 \to R^2$ such that the inputted point is rotated $pi/2$ in counter-clockwise direction about origin. Let P_{near} be a point on J such that distance between origin and P_{near} is minimum. Let P_{far} be a point on J such that distance between origin and P_{far} is maximum. If f(P_{near}) or f(P_{far}) is on J, then the proof is done. Otherwise, we know that f(J) must cross J at some point because f(P_{near}) and f(P_{far}) must be connected by f(J), and there are points on J that are closer to f((P_{near}) than f(P_{far}) by how we defined. Since J is continuous throughout, J and f(J) must intersect at at least one point P. Let P be an ordered pair (x_1, y_1). As seen in the square example, points $(y_1, -x_1)$,P,$(-y_1, x_1)$,$(-x_1, -y_1)$ form a square. ■

4. Parallelograms and rhombuses in Jordan curve

Theorem 4.1. *For a Jordan curve J, let l be any line, then $\exists$ rhombus $P_1P_2P_3P_4$ with $P_1, P_2, P_3, P_4 \in J$ such that $\overline{P_1P_2}$ or $\overline{P_2P_3}$ $\parallel$ to l.*

Theorem 4.1 also implies that every simple closed curve has many parallelograms and rhombuses. Even though it might be straight forward to prove that a Jordan curve contains many parallelograms. But, it is necessary to go over the proof because the technique in proving this basic condition helps us understand the proof of theorem 4.1 better. In figure 5, an example of Jordan curve is shown. Let l be the red line that intersects twice with the curve. Then, we know that we can move a copy of red line in the direction parallel to the red line and obtain another line that has the same length of intersection length to that of red line. The copy of red line is blue-colored line in Figure 5. Furthermore, you can envision

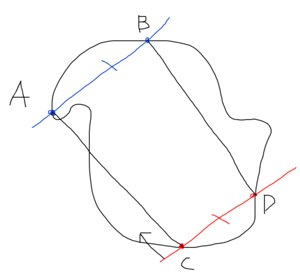


Figure 5. an example of inscribed parallelogram in J

that infinitely distinct lines can be drawn so that they all have different slope and different intersection, which proves that there are infinitely many inscribed parallelograms in a Jordan curve. Now, let's prove that there exists infinitely many rhombuses given any line l.

Proof. If we move the points A and B towards the direction of C and D and move C and D towards the direction of A and B such that $\overline{AB} \parallel \overline{CD}$, $\overline{AB} = \overline{CD}$, and $\overline{AC} = \overline{BD}$. $\overline{AB}$ and $\overline{CD}$ eventually converges to a parallelogram such that $\overline{AB} >> \overline{CD}$. Since we can start out from a parallelogram with $\overline{AB} << \overline{CD}$, we know that at some point, there was a parallelogram with $\overline{AB} = \overline{CD}$, which is a rhombus. Since $\exists$ infinitely many possibilities of lines with length and slope to form rhombuses, there are infinitely many rhombuses in a Jordan curve. ■

An example of how you carry out this process is shown in figure 6. As you can see,

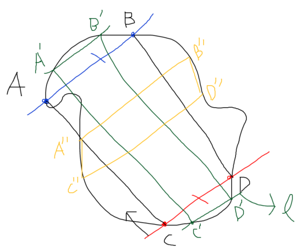


Figure 6. an example of the process of proving the existence of rhombus

A 'B 'C 'D ' is a long parallelogram parallel to l and A "B "C "D " is a wide parallelogram.

5. Inscribed triangle in Jordan curve in two-dimensional space

Theorem 5.1. *For an arbitrary triangle T, $\forall$ Jordan curve J in $\mathbb{R}^2$, $\exists$ three points P_1, P_2, P_3 $\in$ J such that $\triangle P_1P_2P_3 \sim T$.*

Just like a circle circumscribes a triangle similar to an arbitrary triangle, a Jordan curve similarly circumscribes a triangle similar to an arbitrary triangle. This proof uses the idea of dilation, which is stretching and shrinking of a figure about a point.

Proof. Given a random simple closed curve J that is "nice enough", generate a circle C that is entirely containable within J. Also, let an arbitrary triangle be called T. Move the circle in any straight line until a point on a circle is internally tangent to J and without loss of generality name that point X. Let other two points be Y and Z on the circle such that △ XYZ is named in clockwise or counter-clockwise orientation depending on J, which will be defined later, and similar to T. Also, without loss of generality, assume $\overline{YZ}$ is the longest length of $\triangle XYZ$. Then, scale the triangle XYZ to be larger in all three lengths and let the first point that touches J be Y. Select points P and Q on J such that $\overline{PQ}$ is maximized. Keep Y and Z fixed, but move X towards P and dilate Z about Y such that △ XYZ T. If Z happens to be on J, then the proof is complete. However, if Z is not on J, then move Y toward Q while moving Z such that similarity to P is preserved. When $\overline{XY} = \overline{PQ}$, then Z is either outside of J or on J because $\overline{PQ}$ is the longest inscribed line segment on J. If Z is outside of J, then we know at some point during the process of moving Y towards Q, Z was on J. ■

The visual image of the steps of this proof is shown in figures 7,8,9,and 10.

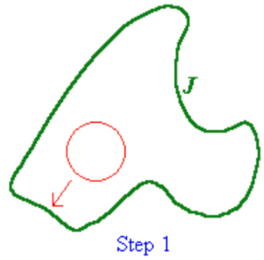


Figure 7. step 1 of proof 2.4

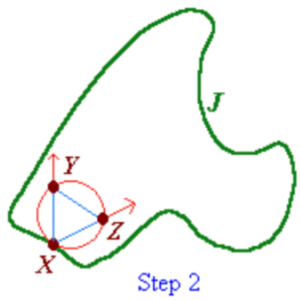


Figure 8. step 2 of proof 2.4

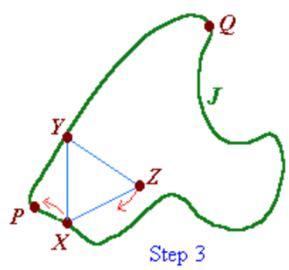


Figure 9. step 3 of proof 2.4

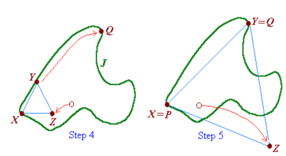


Figure 10. steps 4 and 5 of proof 2.4

6. The density of potential vertex of a triangle in J

Theorem 6.1. *For an arbitrary triangle T, $\forall$ points $P \in$ Jordan curve J, points P,Q,R with Q,$R \in J$ form $\triangle PQR \sim T$.*

This proof is not rigorous in the aspect that it does not include the topological explanation for why every point in J suffices being point fixed point P to create a triangle inscribed in J $\sim T$.

Proof. Let there be an arbitrary triangle T named in the orientation ABC in clockwise orientation. Let there be function f: $\mathbb{R}^2 \to \mathbb{R}^2$ such that f(P) = P, and f(Q) is a point in $\mathbb{R}^2$ such that $\triangle$ PQf(Q) is similar to T with P corresponding to A, B corresponding to Q, and C corresponding to f(Q). Given an arbitrary Jordan curve J and arbitrarily chosen P, if f(J) intersects with J with at least one point which we will call R, then $\triangle PRf(R)$ is similar to T. If f(J) and J intersect at at least one point with any point $t \in$ J, then the set V which includes all points in $t \in$J when t acts as P, f(J) and J intersect at least one point, V is very dense in J. There is a topological argument that V is very dense in J, but I hope it makes intuitive sense from the image that J and f(J) are likely to intersect. ■

The visual steps for this proof is represented in figure 11.

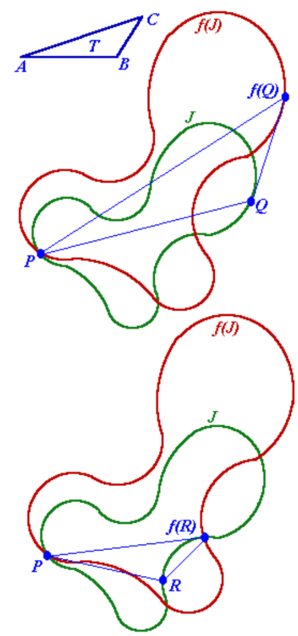


Figure 11. steps for proof 2.5

7. Inscribed triangle in J in three dimensional space

Theorem 7.1. *$\forall$ Jordan curve J in $\mathbb{R}^3$ with point $p \in J$ such that $J'(p) = k$ where k$\in \mathbb{R}$, $\exists$ points $P_1, P_2, P_3 \in J$ such that $\triangle P_1P_2P_3 \sim T$ where T is an arbitrary triangle.*

This proof is the first proof in this paper that requires a visualization of 3-dimensional space. This proof will be easier to understand if you have interacted with Calculus materials before, but if you have not then you only need to know the definition of a function being "smooth". So there is no issue with that! Below is the definition of a function being smooth at one point if you need to know what is means.

Definition 7.2. Function g:$\mathbb{R}^{n-1} \to \mathbb{R}^n$ in space $\mathbb{R}^n$ is smooth at point p $\in$ g if $\exists$ the slope of $\overline{pk}$ and k $\in \mathbb{R}^n$ being selected from one side of p such that $\overline{pk} << 1$ is equal to slope of $\overline{pm}$ where m $\in g$ with g being chosen from other side of k. If you know Calculus, then g is smooth at point P if and only if $g'(P) = k$ k $\in \mathbb{R}$.

Proof. Let T be an arbitrary triangle. Let N be a point on J such that the graph J is differentiable at N. Let P and Q be points on J. When P and Q become arbitrarily close to N, then J is relatively flat near P and Q. ∃ set X in $\mathbb{R}^3$ such that $\triangle PQX$ is similar to T. Let $\overline{PQ}$ be corresponding to the shortest side in T without loss of generality. X becomes a circle linked to J. When P and Q selected so that the distance between two points on J is maximized, then X becomes a circle unlinked from J. When $\overline{PQ}$ moves from being $<< 1$ to maximized in J, we know that at some point in between, X intersects with J. Thus, we can conclude that J in $\mathbb{R}^3$ has an inscribed triangle similar to T. ■

Again, the visualization for this proof is demonstrated in Figure 12. The explanation for "relatively flat" in proof 2.6 is similar to how a differentiable function in $\mathbb{R}^2$ has a straight line tangent to a point, which we can then describe as the function being flat at the point since there exists a tangent line. Similarly, for the Euclidean space $\mathbb{R}^3$, if a function is differentiable at a point, the function has a flat space tangent to that specific point, which we then describe as the function being relatively flat.

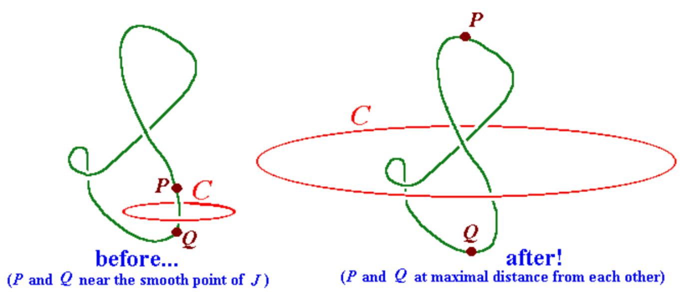


Figure 12. Linked and unliked X

8. Fundamentals of Topology

Informal definition of Surface is the mathematical object that looks like a plane when it is looked at very closely. Formal definition is given below.

Definition 8.1. Surface S is a topological space such that for every point p ∈ S, there is an open set U ⊂ S containing P, and a map f: U → V onto an open subset V ⊂ $\mathbb{R}^2$ so that f is continuous bijection with a continuous inverse.

Torus is informally a surface that is formed by rotating a circle about an axis coplanar to the plane of circle. Formal definition is given below.

Definition 8.2. Torus is the Cartesian product of two unit circles, denoted as $S^1 \times S^1$.

Torus can also be represented by gluing the opposite sides of a unit square as the formal definition is given below.

Definition 8.3. A torus is also defined by the following steps: square S defined by [0,1] × [0,1] when all the points with y value of 1 and all the points with y value of 0 are treated same by gluing those two horizontal sides together to form a horizontal and empty cylinder, then the circles in the leftmost part and rightmost part of the cylinder are glued to treat the points on vertical sides with x=0 and x=1 on S the same.

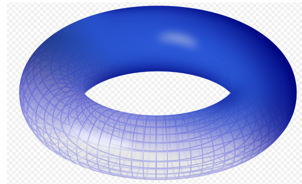


Figure 13. Example of a torus

9. Inscribed rectangle in J

Theorem 9.1. *For Jordan curve J, $\exists$ four points $P_1, P_2, P_3, P_4 \in J$ such that quadrilateral $P_1P_2P_3P_4$ is a rectangle*

To build intuition for this proof, let's visualize how we can glue opposite sides of a square so that a torus is created. Do you see how each ordered pair on the unit square is mapped onto the surface of torus? This means that each point on the surface of Torus maps to an ordered pair on J in $\mathbb{R}^2$. The reason why we glue the opposite sides of square to represent a direct relation to a point on the Jordan curve is because we defined mapping Jordan curve as f:[0,1] $\rightarrow R^n$ with f(0)=f(1). Similar step is done for square to make the function from [0,1] $\times$ [0,1] $\rightarrow R^2$ a one-to-one relation with continuous inverse. Before we get into the proof of theorem 2.7, we need to understand what an ordered pair and unordered pair is.

Definition 9.2. An ordered pair of points in space $\mathbb{R}^n$ is an element of $\mathbb{R}^n \times \mathbb{R}^n$ and distinguishing the difference of order.

Definition 9.3. An unordered pair in space $\mathbb{R}^n$ is an element of $\mathbb{R}^n \times \mathbb{R}^n$ and not distinguishing the difference of order: an unordered pair with same components with different orders are considered the same.

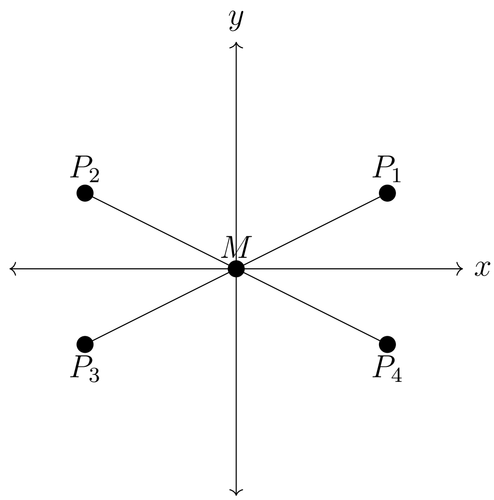
Also, the concept of Cartesian product of sets is used in this proof, so below is the formal definition.

Definition 9.4. Given sets $S_1, S_2, S_3, \ldots S_n$, $S_1 \times S_2 \times S_3 \times \ldots S_n = (s_1, s_2, s_3, \ldots s_n) : s_i \in S_i$, and S^n is defined as n-tuples of each element of S

Lemma 9.5. *When two pairs of points are equal in length and share a common midpoint, the four points form a rectangle.*

For the sake of making the proof of theorem 9.1 easier to understand and ensuring why lemma 9.5 is true, I will prove lemma 9.5 before beginning the proof of theorem 9.1.

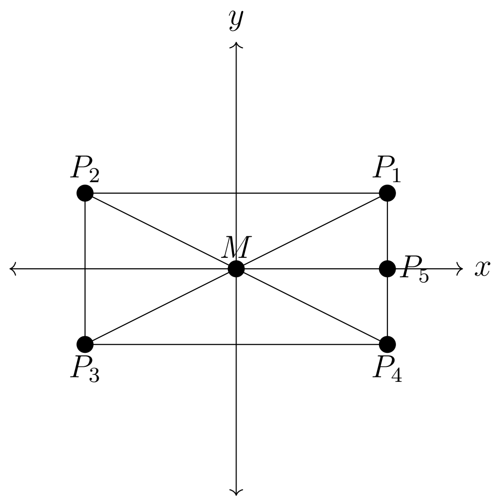
Proof. Let four points be named P_1, P_2, P_3, P_4 such that $\overline{P_1P_3} = \overline{P_2P_4}$. Also, the midpoint M of $P_1, P_3 and P_2, P_4$ be common. Without loss of generality, we can locate points P_1, P_2, P_3, P_4 in $\mathbb{R}^2$ plane such that P_1, P_2, P_3, P_4 are in 1st,2nd,3rd, and 4th quadrant with M being the origin. Below is an example of desired figure.



It is always possible to draw the coordinate system such that the x-axis bisects $\angle P_2MP_3$ and $\angle P_1MP_4$, and the y-axis bisects $\angle P_2MP_1$ and $\angle P_3MP_4$ since the opposite angles formed by two intersecting lines are same by vertical angle theorem. Let P_1 have the coordinate (x_1, y_1) with $x_1, y_1 > 0$ by the assumption we made at the beginning of the proof. Then, P_3 must have the coordinate $(-x_1, -y_1)$ because of the midpoint formula so that M is the origin. Let P_2 have the coordinate (x_2, y_2) and $x_2 < 0$ and $y_2 > 0$ by the assumption also. Then P_4 must have the coordinate $(-x_2, -y_2)$ because of the midpoint formula again. Since we set the condition to make $\overline{P_1P_3} = \overline{P_2P_4}$, by the distance formula,

$$x_1^2 + y_1^2 = x_2^2 + y_2^2 \tag{9.1}$$

From equation 9.1, we see that $\overline{P_1M} = \overline{P_4M}$ because $\overline{P_1M} = x_1^2 + y_1^2$ and $\overline{P_4M} = x_2^2 + y_2^2$. When we connect P_1 and P_4, we see that $\triangle P_1MP_4$ is an isosceles triangle.Let the intersection of $\overline{P_1P_4}$ with x-axis be P_5. Because of one of the isosceles triangles properties, $\angle MP_4P_1 = \angle MP_1P_4$. By ASA similarity, $\triangle P_1MP_5 \cong \triangle P_4MP5$. This proves that $\overline{P_1P_4}$ is perpendicular to x-axis at the point of intersection. By symmetry about origin M, we see that $\overline{P_2P_3} = \overline{P_1P_4}$. Same process can be applied to points P_1 and P_2 as well as the opposite points P_3 and P_4 because of the same component of the ordered pairs with differing signs to prove that $\overline{P_1P_2} = \overline{P_3P_4}$. Since opposite sides are equal in a quadrilateral, we prove that quadrilateral $P_1P_2P_3P_4$ is a rectangle.



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Now that we have proven lemma 9.5, we will begin proving theorem 9.1. The visuals to follow while reading the proof are in figures 14, 15, and 16 below.

Proof. Without loss of generality and for simplicity purposes, let square S be defined as [0,1] × [0,1]. Let g be a one-to-one function that takes in two points (x_1,y_1),(x_2,y_2) ∈ Jordan curve J and map to a point (x_3,y_3,z_3) in $\mathbb{R}^3$ such that (x_3,y_3,z_3) is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ above (($x_1 + x_2$)/2,($y_1 + y_2$)/2) in orthogonal direction to $\mathbb{R}^2$. Let u and v be points ∈ J. Since g(u,v)=g(v,u), it is only necessary to consider the unordered pair of points on square. Let unordered pairs on square be represented by set D=(t,s) ∈ [0,1] × [0,1]: $t <= s$. Let function γ be: [0,1] ∈ S → J. Since γ (0)=γ(1), $\gamma(D) \times \gamma(D)$ is topologically a Möbius strip, where the edge of the strip is f($\gamma(\delta)$,$\gamma(\delta)$: $\delta \in$ [0,1])=$(\gamma_x(s), \gamma_y(s), 0)$ since dist (J(s),J(s))=0. We will omit proof for why the Möbius strip intersect itself in $\mathbb{R}^3$ but hopefully it makes intuitive sense, especially visualizing the g ∘ J in the image below. And since there is an intersection point in $\mathbb{R}^3$ of the Möbius strip, we know that two unordered pairs of J have same value when plugged into g. This means that there are two pairs of points that are equidistant and share a mid point on J, so a rectangle is inscribed in J by lemma 9.5. ■

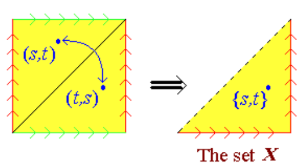


Figure 14. Folding across the diagonal

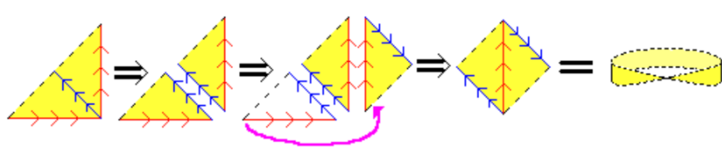


Figure 15. Making Mobius strip by gluing and twisting the square

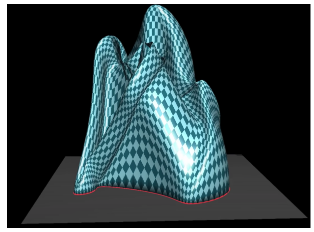


Figure 16. How Mobius strip can map onto 3-dimensional space

10. Recent Progress in Square Peg problem

Now that we have learned about various characteristics of Jordan curve. We will now discover some more recent progress toward the square peg problem. Well, it is not really about Jordan curves circumscribing a square, but rather it is about continuous or differentiable Jordan curve's ability to describes different more various sets of shapes like cyclic quadrilateral, isosceles trapezoids, and more. The sections to come after this come from works of Matschke,Tao, Karasev, Toeplitz, and many more mathematicians. Let's define some useful definitions that will be useful in reading upcoming sections.

Definition 10.1. Let δ be the set of points $\in \mathbb{R}^2$ and located internally of J and including the points on J. If the set of points γ in the closed loop formed by the expression below

$$\bigcup_{i=1}^{\infty}(y = f\,'(x_i)(x - x_i) + y_i)$$

Where $(x_i, y_i) \in J$ and $(x, y) \in R^2$ and $\gamma \cap \delta = \delta$, then J is a convex Jordan curve.

Definition 10.2. C^n is the class of differentiable graphs whose nth derivative is continuous.

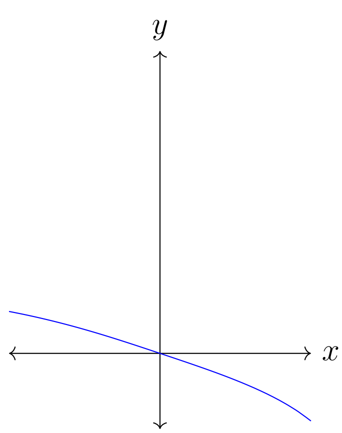
To give an intuition for definition 10.2, circle is an example of a graph that belongs in the class C^∞. Let the circle C be defined by the equation below:

$$x^2 + y^2 = k^2 \tag{10.1}$$

where k $\in \mathbb{Z}$. When C is differentiated with respective to x, y=C' has the equation:

$$y = -x/(k^2 - x^2)^{1/2} \tag{10.2}$$

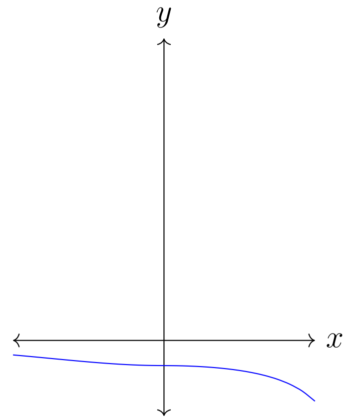
And a part of equation 10.2 is graphed below.



When C' is differentiated with respective to x again, y=C'' has the equation below:

$$y = -k^2/(k^2 - x^2)^3/2. \tag{10.3}$$

And again, a part of equation 10.3 is graphed below.



As you can see, all the graphs of the equations 10.2 and 10.3 are seemingly smooth and differentiable at all points. So, C^∞ means that no matter how many times the equation of circle is differentiated, the differentiated equation will always be continuous and smooth. Moving onto more important parts of the work by Matschke, it is important to note that the cyclic quadrilaterals are the most general case of sets of quadrilaterals out of squares, rectangles, and isosceles trapezoids. Let ζ_S represent the set of squares, ζ_R represent the set of rectangles, ζ_I represent the set of isosceles trapezoids, and ζ_C represent the set of cyclic quadrilaterals. The relationship above can be mathematically represented as

$$\zeta_S \subseteq \zeta_R \subseteq \zeta_I \subseteq \zeta_C$$

Definition 10.3. J^k_{conv} denotes k-times differentiable convex Jordan curve where k $\in \mathbb{Z}$.

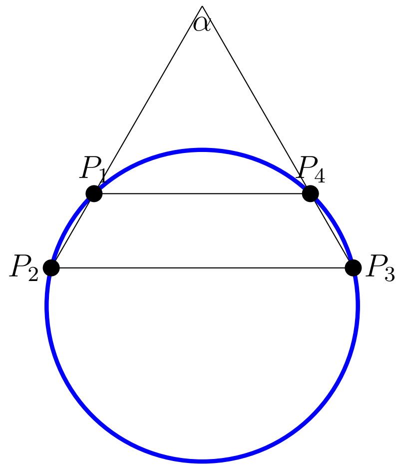
Matschke states both theorems below and aims to give insights to those proofs using Karasev and Tao's signed area argument.

Theorem 10.4. *J^0_{conv} inscribes the set ζ_I, and ζ_I is the largest possible set that J^0_{conv} can inscribe.*

Theorem 10.5. *The class J^1_{conv} inscribes the set ζ_C,and ζ_C is the largest set that J^1_{conv} can inscribe.*

Theorem 10.6. *generalization: if the angle that the Jordan curve J makes at a singular point is larger than the min(α, β), then J circumscribes the cyclic quadrilateral Q*

α and β are the angles formed by extending the opposite sides of Q until the point of intersection. In the case of J^1_{conv}, the angle condition from the generalization is never satisfied since no angle is made at a point, so theorem 2.9 is trivially satisfied. When Q is an isosceles trapezoid as the figure below shows, the min(α, β)=0, as $\angle\beta$ is never formed in the figure. So, J^0_{conv} always circumscribes an isosceles trapezoid.



For theorem 10.5, let Q=$P_1P_2P_3P_4$ be a cyclic quadrilateral. A triangle similar to $\triangle P_1P_2P_3$ is chosen on J. Then, the signed area argument with proof by contradiction is used to prove that the fourth point P_4 must be on J. For theorem 10.4, the possible complex structure of Jordan curve J due to restriction of only having to be continuous rather than differentiable allows an easier approach to determining even lower and upper bounds of ζ_I. Since the extension of what Matschke does to provide meaningful insights in proving theorem 10.5 and theorem 10.4 is not shown in this paper, please read [2] to learn more about those theorems.

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[1] [2] [3] [4] [5] [6]