Introduction to Hyperplane Arrangements

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Euler Circle

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Outline

1. Preliminaries
2. Operations of hyperplane arrangements
3. Counting regions
4. Using finite fields

Introduction

Hyperplanes

Definition 1.1

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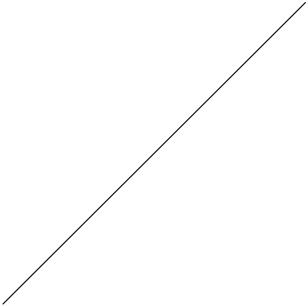

Hyperplane Arrangements

Definition 1.2

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$$\mathcal{A} = \{H_0, H_1, \cdots H_m\}$$

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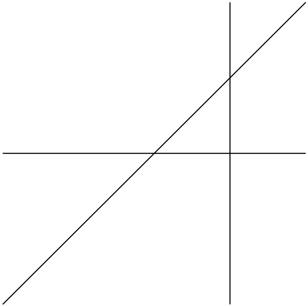

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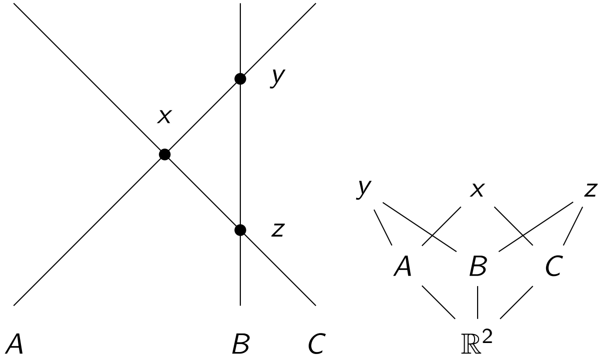
- The poset is ordered by reverse-inclusion meaning $x \leq y$ if $x \supseteq y$

The Intersection Poset



The Mobius function

The Mobius function μ is defined recursively on the elements of $L(\mathcal{A})$

$$\sum_{x \leq z} \mu(x) = 0$$

for all elements $z \in L(\mathcal{A})$ and $\mu(\mathbb{R}^n) = 1$.

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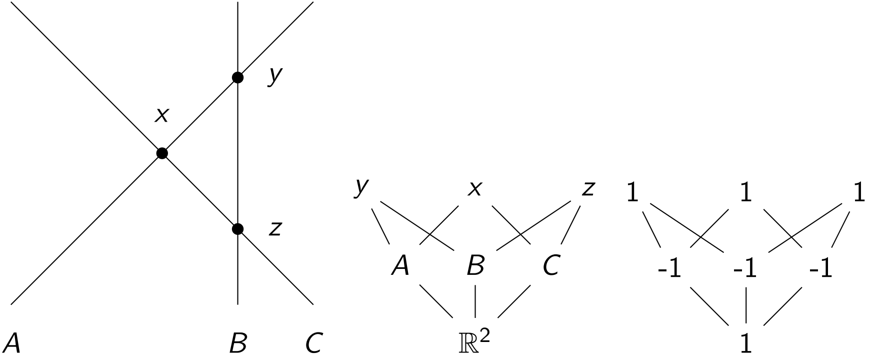

Characteristic Polynomial

The characteristic polynomial of an arrangement $\mathcal{A}$ is defined as

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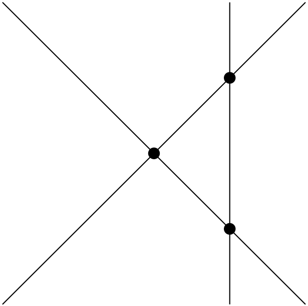

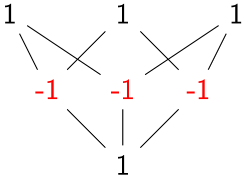

$t^2 - 3t + 3$

General position

We will mostly be dealing with arrangements in general position.

Definition 1.3

$\mathcal{A}$ is in general position if

$$\{H_1, \ldots, H_p\} \subseteq \mathcal{A}, p \leq n \Rightarrow \dim(H_1 \cap \ldots H_p) = n - p$$

$$\{H_1, \ldots, H_p\} \subseteq \mathcal{A}, p > n \Rightarrow H_1 \cap \ldots H_p = \emptyset.$$

You should be able to slightly move around the hyperplanes and have the same number of regions.

Deletion-Restriction

Definitions

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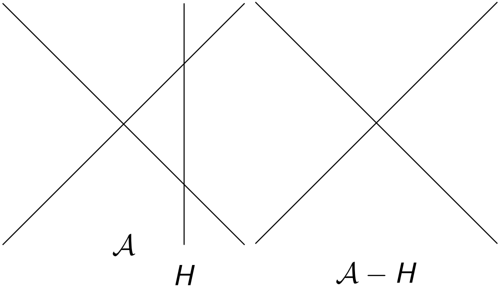

Restriction

Definition 2.2 (Restriction)

$\mathcal{A}/H$ represents the new arrangement with only elements greater than or equal to H in the $L(\mathcal{A})$.

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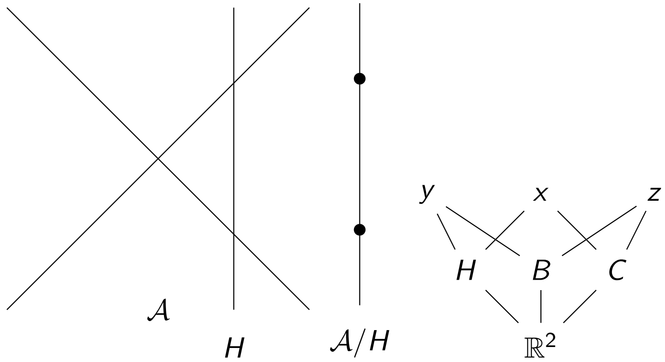

Theorem about regions

Theorem 2.3

Given an arrangement $\mathcal{A}$ and a hyperplane H in $\mathcal{A}$,

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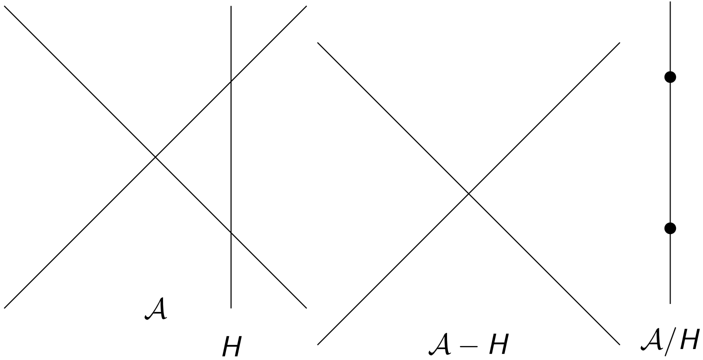
- Consider the regions that are cut by H

Theorem about regions



Characteristic polynomials

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Given an arrangement $\mathcal{A}$ and a hyperplane H in $\mathcal{A}$,

$$\chi_{\mathcal{A}}(t) = \chi_{\mathcal{A}-H}(t) - \chi_{\mathcal{A}/H}(t).$$

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- Proof uses Whitney's theorem

Zaslavsky's Theorem

Theorem 2.5

$$r(\mathcal{A}) = (-1)^n \chi_{\mathcal{A}}(-1)$$

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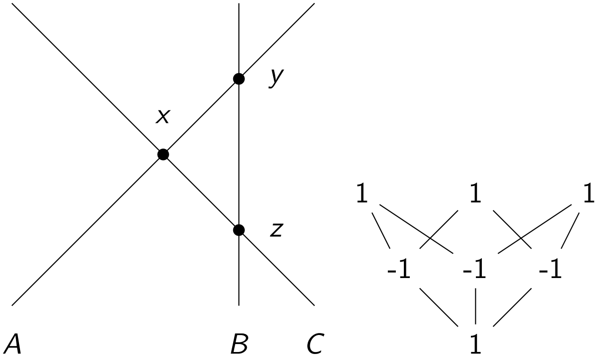
Proof uses the fact that both sides can be broken down into $\mathcal{A} - H$ and $\mathcal{A}/H$ terms. Induction can then be used.

Zaslavsky's Theorem



Finite Field Method

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Theorem 3.1

For a sufficiently large prime q where $L(\mathcal{A}) \cong L(\mathcal{A}_q)$,

$$\chi_{\mathcal{A}}(q) = \#(\mathbb{F}_q^n - \bigcup_{H \in A_q} H)$$

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1. Möbius Inversion

Examples

We will be able to use the finite field method to easily calculate the number of regions for Coxeter arrangements.

$$\mathcal{A}_n = \{x_i - x_j = 0 : 1 \leq i < j \leq n\}$$

$$\mathcal{C}_n = \{x_i - x_j = -1, 0, 1 \ : \ 1 \leq i < j \leq n\}$$

$$\mathcal{D}_n = \mathcal{A}_n \cup \{x_i + x_j = 0 : 1 \leq i < j \leq n\}$$

$$\mathcal{B}_n = \mathcal{D}_n \cup \{x_i = 0 : 1 \leq i \leq n\}$$

$\mathcal{A}_n$

- How many ways are there to pick a coordinate $(x_1, x_2, \cdots x_n)$ such that $x_i \neq x_j$.

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- $|\chi_{\mathcal{A}_n}(-1)| = n!$

$\mathcal{A}_3$

Conclusion

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- Graphical arrangements

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- When does the characteristic polynomial completely factor over the integers?