HYPERPLANE ARRANGEMENTS

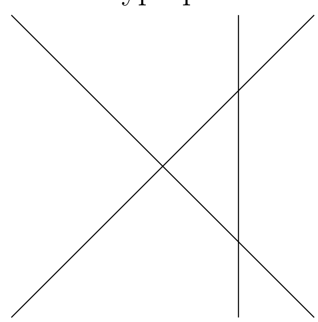
VINCENT CHENG

ABSTRACT. This paper introduces basic components of hyperplane arrangements and combinatorial invariants for each arrangement such as the intersection poset and the characteristic polynomial. The primary goal in the study of hyperplane arrangements is the find the number of regions given an arrangement, so we introduce the Möbius function and how it can be utilized to count regions. We also introduce the finite field method which provides an alternative way to find the number of regions for Coxeter arrangements.

1. INTRODUCTION

Definition 1.1. A **hyperplane arrangement** $\mathcal{A} = \{H_1, H_2, \cdots H_m\}$ is the set of hyperplanes in K^n where K is a field and each hyperplane H_i has dimension $n-1$.

We will only consider $K = \mathbb{R}$ for this paper. Below is an example of a hyperplane arrangement in two dimensions with two hyperplanes.



There are some broad categories that we can classify our arrangements in which will be very helpful for us.

Definition 1.2. A hyperplane arrangement, $\mathcal{A}$, is **central** if

$$\bigcap_{H \in \mathcal{A}} H \neq \emptyset$$

Definition 1.3. An arrangement, $\mathcal{A}$, is defined to be in **general position** if

$$\{H_1, \ldots, H_p\} \subseteq \mathcal{A}, p \leq n \Leftrightarrow dim(H_1 \cap \ldots H_p) = n - p$$

$$\{H_1, \ldots, H_p\} \subseteq \mathcal{A}, p > n \Leftrightarrow H_1 \cap \ldots H_p = \emptyset$$

and otherwise affine.

In casual terms, a general arrangement is an arrangement where you can move around any of the hyperplane by a "small" amount and have the resulting arrangement preserve all the qualities of the original one. For example, a general arrangement in three dimensions won't have any three lines intersect at a common point. Our example above is an arrangement in general position.

Date: July 11, 2022.

Definition 1.4. A **region** of $\mathcal{A}$ in $\mathbb{R}^n$ is defined to be a connected component of

$$\mathbb{R}^n - \bigcup_{H \in \mathcal{A}} H$$

and $r(\mathcal{A})$ denotes the number of regions in $\mathcal{A}$.

As a warm up, we can try our hand at finding the number of regions for the braid arrangement which is defined as

$$\mathcal{B}_n : x_i - x_j = 0, \;\; 1 \le i < j \le n.$$

Given a point $(a_1, \cdots, a_n)$, we can determine which side hyperplane $x_i - x_j = 0$ by considering whether $a_i > a_j$ or $a_i < a_j$. Hence, we can see that each ordering of $(a_1, \cdots, a_n)$ describes a different region. Hence $r(\mathcal{B}_n) = n!$. We can see an example of $n = 3$ where $r(\mathcal{B}_3) = 6$ and the orderings are

$$x_1 < x_2 < x_3, \quad x_1 < x_3 < x_2, \quad x_2 < x_1 < x_3, \quad x_2 < x_3 < x_1, \quad x_3 < x_1 < x_2, \quad x_3 < x_2 < x_1.$$

It is not always this easy to find the number of regions using a combinatorial argument. We would like to develop some more tools to help us. Regions are formed when areas in space are separated by hyperplanes and their intersections. Hence, it's natural to assign an intersection poset to each arrangement.

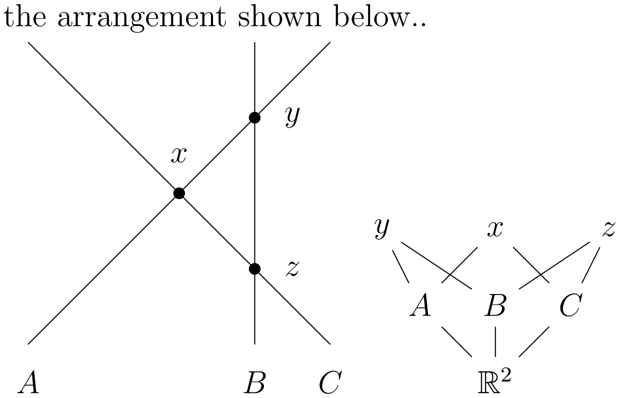
Definition 1.5. A poset, P is a set of elements with a relation $\le$ such that for every pair of elements x, y in P,

(1) $x \le x$
(2) If $x \le y$ and $y \le x$, $x = y$
(3) If $x \le y$ and $y \le z$, $x \le z$

Hasse diagrams are a good way to visualize posets and the relationship between their elements. The intersection poset of arrangement $\mathcal{A}$ will be denoted $L(\mathcal{A})$ and consist of:

(1) the entire space $\mathcal{A}$ is in
(2) each individual hyperplane
(3) the intersections of the hyperplanes

with the relation of reverse-inclusion meaning $x \le y$ if $x \supseteq y$. It can be observed that the higher up you go on the intersection poset, the lower the dimension of the elements are. The minimal element is always the entire space, $\mathbb{R}^n$. Here we provide the intersection poset and the Hasse diagram for the arrangement shown below..



Definition 1.6. A lattice is a poset where every two elements have a join and a meet (every two elements have a least upper bound and a greatest lower bound).

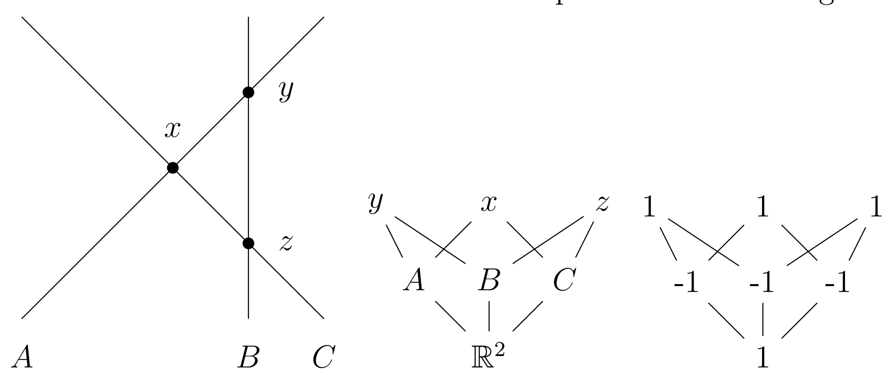
It is not hard to deduce that any two elements of the intersection poset will have a greatest lower bound since every element shares an element smaller than them, namely $\mathbb{R}^n$. Every element will have a least upper bound if there exists an element that is greater than every other element (a point of intersection between every hyperplane). Hence, if an arrangement is central, it's intersection poset will be a lattice. We want to do operations on the intersection poset to somehow find the number of regions, hence we define the Möbius function which takes an element from the poset and returns an integer value.

Definition 1.7. The Möbius function denoted $\mu : L(\mathcal{A}) \to \mathbb{Z}$ is defined recursively as

$$\sum_{z \leq x} \mu(z) = 0$$

and $\mu(\hat{0}) = 1$ where $\hat{0}$ is the smallest element of the poset ($\mathbb{R}^n$).

This is a specific definition of the general Möbius function on the intersection poset with $\mu(\hat{0}, x)$. We rarely use the general case of the Möbius function, but we will define the general version of the Möbius function later on when we introduce Möbius inversion. For now, this definition will suffice. The Möbius values of the example we've been using is shown below.



For every arrangement, there is a characteristic polynomial which combines the Möbius values of every element. We will see later how this is extremely powerful for finding regions.

Definition 1.8. The characteristic polynomial of an arrangement $\mathcal{A}$ is defined to be

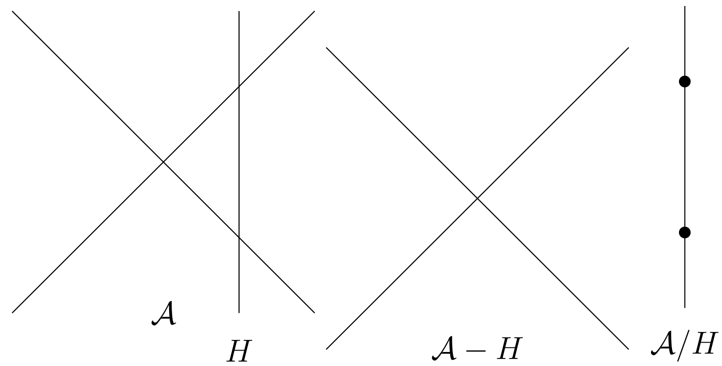
$$\chi_{\mathcal{A}}(t) = \sum_{x \in L(\mathcal{A})} \mu(x) \cdot t^{\dim(x)}$$

The characteristic polynomial of the arrangement above is $t^2 - 3t + 3$.

2. Deletion-Restriction

Given an arrangement and a specific plane in the arrangement, there are two interesting variants of the arrangement, namely deletion and restriction. These variants are interesting as they will give us a really simple proof of Zaslavsky's theorem is a powerful theorem that will be shown later.

Definition 2.1. Given an arrangement $\mathcal{A}$ and a hyperplane $H \in \mathcal{A}$, the *deletion* of H,$\mathcal{A} - H$, is defined to the the new arrangement where H is removed. The *restriction* of a hyperplane in an arrangement, $\mathcal{A}/H$ is making a new intersection poset that only includes elements greater than or equal to H.



Some sources will write $(\mathcal{A}, \mathcal{A} - H, \mathcal{A}/H)$ as $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ which is just different notation. We will now discuss some interesting properties that arise out of these variants.

Theorem 2.2.

$$r(\mathcal{A}) = r(\mathcal{A} - H) + r(\mathcal{A}/H)$$

Proof. We can first look at the regions that are unaffected by H. These regions are not changed so they are still present in $\mathcal{A} - H$ and don't exist in $\mathcal{A}/H$. Regions that are cut by H go from two regions to one in $\mathcal{A}$ to $\mathcal{A} - H$. However, these regions all exist in $\mathcal{A}/H$ and are added back. ■

Looking at the example above, we can see that $r(\mathcal{A}) = 7$, $r(\mathcal{A} - H) = 4$, and $r(\mathcal{A}/H) = 3$. Before we go into the next theorem, we can see from [S$^+$04] that another way to define the characteristic polynomial is

$$\chi_{\mathcal{A}}(t) = \sum_{\substack{\mathcal{B} \subseteq \mathcal{A} \\ \mathcal{B} \text{ central}}} (-1)^{\#(\mathcal{B})} t^{n - \operatorname{rank}(\mathcal{B})}.$$

This is called Whitney's theorem and the reader should refer to [S$^+$04] for more details about this.

Theorem 2.3. *Given the previous definitions of $\mathcal{A} - H$ and $\mathcal{A}/H$,*

$$\chi_{\mathcal{A}}(t) = \chi_{\mathcal{A}-H}(t) - \chi_{\mathcal{A}/H}(t)$$

Proof. We prove this result by splitting up the sum from Whitney's Theorem into two parts depending on whether $H \subseteq B$ or $H \nsubseteq B$. We split our sum into

$$\chi_{\mathcal{A}}(t) = \sum_{\substack{H \notin \mathcal{B} \subseteq \mathcal{A} \\ \mathcal{B} \text{ central}}} (-1)^{\#\mathcal{B}} t^{n - \operatorname{rank}(\mathcal{B})} + \sum_{\substack{H \in \mathcal{B} \subseteq \mathcal{A} \\ \mathcal{B} \text{ central}}} (-1)^{\#\mathcal{B}} t^{n - \operatorname{rank}(\mathcal{B})}.$$

The first term is quite easy to deal with as by Whitney's theorem, we know that

$$\sum_{\substack{H \notin B \\ \mathcal{B} \text{ central}}} (-1)^{\#\mathcal{B}} t^{n - \operatorname{rank}(\mathcal{B})} = \chi_{\mathcal{A}-H}(t).$$

What's left to prove is that the second term equals $-\chi_{\mathcal{A}/H}(t)$. For each subarrangement $\mathcal{B}$ that contains H, define $\mathcal{B}_1$ to be $\mathcal{B}/H$. We can see that $\mathcal{B}_1$ is still central and it is also a subarrangement of $\mathcal{A}/H$. In fact, we can see that the $\mathcal{B}_1$'s describe all the subarrangements of $\mathcal{A}/H$. Hence, we get that $\#\mathcal{B}_1 = \#\mathcal{B} - 1$ and $\text{rank}(\mathcal{B}_1) = \text{rank}(\mathcal{B}) - 1$. With the these observations, we get

$$\sum_{\substack{H\in\mathcal{B}\subseteq\mathcal{A}\\ \mathcal{B}\text{ central}}} (-1)^{\#\mathcal{B}} t^{n-\text{rank}(\mathcal{B})} = \sum_{\mathcal{B}_1\subseteq\mathcal{A}/H} (-1)^{\#\mathcal{B}_1+1} t^{(n-1)-\text{rank}(\mathcal{B})} = -\chi_{\mathcal{A}/H}(t).$$

The second equality holds because we can take the +1 in the exponent out of the sum to get a (-1) term which gives us the negative as desired and the space of $\mathcal{A}/H$ is $(n-1)$ is the exact form of Whitney's theorem with $\mathcal{A}/H$ as the arrangement. This finishes the proof as it shows that

$$\chi_{\mathcal{A}}(t) = \chi_{\mathcal{A}-H}(t) - \chi\mathcal{A}/H(t).$$

■

3. Möbius Inversion

We begin by redefining a more general Möbius function

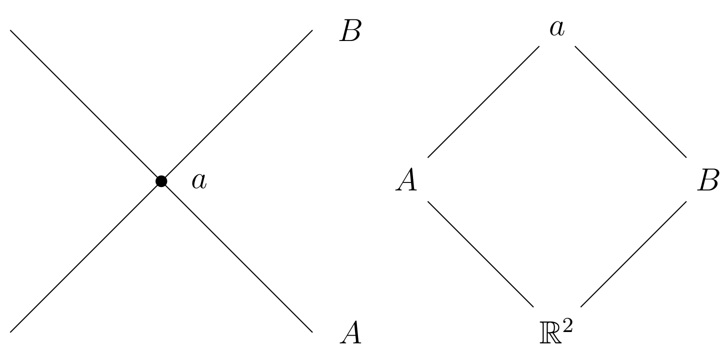
Definition 3.1. The Möbius function, $\mu : P \to \mathbb{Z}$, takes in two elements from the intersection poset and returns an integer value.

(1) $\mu(x, x) = 1$ for all $x \in P$
(2) $\mu(x, y) = -\sum_{x\le z<y} \mu(x, z)$ for all $x < y \in P$. which can also be written as

$$\sum_{\substack{z\\ x\le z\le y}} \mu(x, z) = 0$$

(3) $\mu(x, y) = 0$ if $x \nleq y$

Through this definition, we can see that our previous definition only described the specific case where we let $\mu(x) = \mu(\hat{0}, x)$. It is hard to see what exactly this function does so we will provide an example arrangement with all $\mu(x, y)$ values in a matrix.



$$\mu(\text{row}, \text{col}) = \begin{array}{c} \\ \hat{0} \\ A \\ B \\ a \end{array} \begin{array}{c} \begin{array}{cccc} \hat{0} & A & B & a \end{array} \\ \begin{pmatrix} 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

The reader should take some time to think about why these are the correct values to gain a deeper familiarity with the Möbius function as it will be used a lot later on.

Definition 3.2. Define the function $\zeta : P \to \mathbb{Z}$ as

$$\begin{cases} \zeta(x,y) = 1 & x \leq y \\ \zeta(x,y) = 0 & \text{otherwise} \end{cases}$$

We will once again use the arrangement above to illustrate the values of ζ on a poset.

$$\zeta(\text{row}, \text{col}) = \begin{array}{c} \\ \hat{0} \\ A \\ B \\ a \end{array} \begin{array}{c} \begin{array}{cccc} \hat{0} & A & B & a \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Even though it seems that these two functions are unrelated, we will now prove why they are inverse functions.

Lemma 3.3. *ζ is the inverse of μ meaning*

$$\zeta\mu = I = \mu\zeta$$

where I:

$$\begin{cases} I(x,y) = 1 & x = y \\ I(x,y) = 0 & \textit{otherwise} \end{cases}$$

Proof. We define $\zeta\mu$ as the matrix multiplication of ζ and μ meaning

$$\zeta\mu(x,y) = \sum_{z \in P} \mu(x,z)\zeta(z,y).$$

which element-wise multiplies the x-row of μ and the y-column of ζ then sums the results to get one value of I. It suffices to show that the resulting matrix we get is the identity matrix. We know for all $z > y, \zeta(z,y) = 0$, so those terms disappear. The other values of $\zeta(z,y) = 1$ so we can turn our attention to μ now. What is left is

$$\sum_{z \leq y} \mu(x,z).$$

We also know that, by definition, when $x \nleq z, \mu(x,z) = 0$. Hence, what is left is

$$\sum_{\substack{z \\ x \leq z \leq y}} \mu(x,z).$$

If $x = y$, this value will be 1, and if $x < y$ this value will be 0, both by the definition of the Möbius function. If $y < x$ this value will be 0 because there are no terms to sum. Hence showing that the resulting product is the identity matrix which was what we desired. ∎

Now we will give the Möbius inversion formula which will be extremely useful for proving other theorems further on in this paper.

Theorem 3.4. *Given functions $f, g : P \to \mathbb{Z}$, we have*

$$f(x) = \sum_{x \leq y} g(y) \Leftrightarrow g(x) = \sum_{x \leq y} \mu(x,y) f(y)$$

Proof. We can prove this using ideas of matrix multiplication. Define $[\mu], [\zeta]$ as the $n \times n$ matrices shown above and envision f and g as column vectors with each entry being the value of an element of the poset passed through the respective function. Notice that the LHS of the equation on the left is equivalent to the matrix multiplication of

$$\vec{f} = [\zeta]\vec{g}$$

since terms with $x < y$ are 0 in $[\zeta]$ while terms with $x \geq y$ are 1. This means doing the matrix multiplication is equivalent to summing up the terms greater than x. Now we can remember that $[\mu][\zeta] = I$ to get the following steps

$$\vec{f} = [\zeta]\vec{g} \Leftrightarrow [\mu]\vec{f} = [\mu][\zeta]\vec{g} \Leftrightarrow [\mu]\vec{f} = \vec{g}\,.$$

Converting the last equation away from vector form, we get $g(x) = \sum_{x \leq y} \mu(x, y) f(y)$ since $\mu(x, y) = 0$ for $x > y$ as desired. ■

4. Weisner's Theorem

Weisner's theorem was originally used as a tool to prove the later Zaslavsky's theorem in [Sam03], however, we've developed an easier approach for that proof. We will still go over this proof as it's an interesting result by itself.

We define a mapping $f : x \to x \vee p$ for any $x \in L$ and a fixed $p \in L$ where L is the intersection poset. Here $\vee$ represents the join of two elements which means the least upper bound.

Definition 4.1. $\bar{L}$ is the image of f on a lattice L.

Another way to think of the image is that $\bar{L}$ contains all $y \in L$ such that $y \geq p$.

Theorem 4.2. *For all* $z \in \bar{L}$*,*

$$\sum_{\substack{x \in L \\ x \vee p = z}} \mu(\hat{0}, x) = \begin{cases} 0 & \textit{if } p > 0 \\ \mu(\hat{0}, z) & \textit{if } p = 0 \end{cases}$$

Proof. We can first see that $p = 0$ is the trivial case. If $p = 0$, $L = \bar{L}$. Also, the only x that will satisfy the conditions of the summand will be z since only $z \vee \hat{0} = z$, hence

$$\sum_{\substack{x \in L \\ x \vee \hat{0} = z}} \mu(\hat{0}, x) = \mu(\hat{0}, z)$$

Now we will consider the case where $p > \hat{0}$. Define

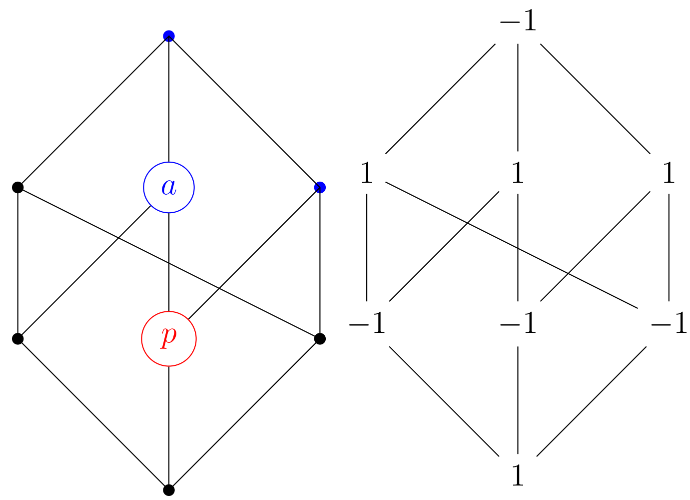
$$g(z) = \sum_{\substack{x \in L \\ x \vee p = z}} \mu(0, x)$$

$$f(z) = \sum_{\substack{y \in \bar{L} \\ y \leq z}} g(y)$$

Our goal in defining f is to later use the Möbius inversion formula to show that $g(z) = 0$ for all $p > 0$. We can rewrite f as

$$f(z) = \sum_{\substack{y \in \bar{L} \\ y \leq z}} \sum_{\substack{x \in L \\ x \vee p = y}} \mu(\hat{0}, x)$$

We will first look at what f is doing through a concrete example. Below is the intersection poset of a hypothetical arrangement where choose the red node as p and the blue nodes represents the elements in $\hat{L}$.



$$f(z) = g(z) + g(p) = (1-1) + (-1+1) = 0$$

We see that f actually simplifies to

$$f(z) = \sum_{\substack{x \in L \\ x \leq z}} \mu(\hat{0}, x).$$

So $f(z)$ is summing over the Möbius values of $[\hat{0}, z]$ which by definition gives us

$$f(z) = 0 \ \ \forall\, z > \hat{0}$$

$$f(z) = 1 \ \text{ if } z = \hat{0}.$$

Now we can use the Möbius inversion formula on f and g to get

$$g(z) = \sum_{\substack{y \in \bar{L} \\ y \leq z}} \mu(y, z) f(y)$$

If $p > 0$, we know that $y > \hat{0}$ for all y since $y \geq p$ from $y = x \vee p$. However, this also means $f(y) = 0$ for all y which gives us

$$g(z) = 0 \text{ if } p > \hat{0}$$

If $p = 0$, the only non-zero term in the summand of $g(z)$ is when $y = \hat{0}$ which gives us

$$g(z) = \mu(\hat{0}, z) \text{ if } p = 0$$

This finishes the proof of Weisner's theorem.

■

5. Zaslavsky's Theorem

This is one of the most powerful theorems in calculating the number of regions in an arrangement. We will use some of the previous theorems that we have developed.

Theorem 5.1 (Zaslavsky). *Given an arrangement, $\mathcal{A}$,*

$$r(\mathcal{A}) = (-1)^n \chi_{\mathcal{A}}(-1).$$

An equivalent way of saying this is that the number of regions in an arrangement is equal to the absolute value of the Möbius value of each element in the intersection poset.

Proof. We will prove this using strong induction on the number of hyperplanes. Consider the base case $\emptyset$ where there are not any hyperplanes. It is obvious that there is 1 region for this arrangement. $L(\emptyset)$ contains only one element $\mathbb{R}^n$ and $\chi_\emptyset(t) = t^n$. So we get

$$r(\emptyset) = (-1)^n \cdot (-1)^n = 1$$

which means we're done with this case.

For our inductive hypothesis, we assume that this theorem holds for all arrangements with less than n hyperplanes. Then, we know that

$$r(\mathcal{A}) = r(\mathcal{A}-H)+r(\mathcal{A}/H) = (-1)^n\chi_{\mathcal{A}-H}(-1)+(-1)^{n-1}\chi_{\mathcal{A}/H}(-1) = (-1)^n(\chi_{\mathcal{A}-H}(-1)-\chi_{\mathcal{A}/H}(-1)).$$

This should ring some bells as from Theorem 3.4, we know that

$$\chi_{\mathcal{A}}(t) = \chi_{\mathcal{A}-H}(t) - \chi_{\mathcal{A}/H}(t)$$

which gives us

$$r(\mathcal{A}) = (-1)^n(\chi_{\mathcal{A}-H}(-1) - \chi_{\mathcal{A}/H}(-1)) = (-1)^n\chi_{\mathcal{A}}(-1)$$

as desired. ∎

6. The Finite Field Method

Now we will introduce the finite field method which gives us a new method to calculate the characteristic polynomial for specific types of arrangements. A finite field $\mathbb{F}_q$ where $q = p^r$ for some prime p is a field containing q elements. For our purposes, we can consider $\mathbb{F}_q$ to be the numbers possible mod q. For example. $\mathbb{F}_5^2$ can be thought of as a non-continuous 5×5 grid of points with coordinates $(0,0) \cdots (4,4)$. The following theorem is the cornerstone to all computations with the finite field method. We define $\mathcal{A}_q$ to be the new arrangement where the coefficients of each hyperplane are taken mod q. Note that it is not always true that $L(\mathcal{A}) \cong L(\mathcal{A}_q)$, and when it is true, we say $\mathcal{A}$ has a good reduction mod q.

Theorem 6.1. *For a sufficiently large prime q where $L(\mathcal{A}) \cong L(\mathcal{A}_q)$,*

$$\chi_{\mathcal{A}}(q) = \#(\mathbb{F}_q^n - \bigcup_{H \in A_q} H)$$

Example. We can consider the example in two dimension where we have

$$\mathcal{A} = \{H_0 : x_2 = 2x_1, H_1 : x_2 = -2x_1.\}$$

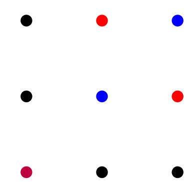
The intersection poset would look like

$$\begin{tikzcd}[column sep=small] & (0,0) \arrow[dl, dash] \arrow[dr, dash] & \\ H_0 \arrow[dr, dash] & & H_1 \arrow[dl, dash] \\ & \mathbb{R}^2 & \end{tikzcd}$$

We would also have

$$\chi_{\mathcal{A}}(t) = t^2 - 2t + 1.$$

We can draw out $\mathbb{F}_3^2$ with H_0 represented as the red points, H_1 represented as the blue points, and $(0,0)$ represented by the purple point.



Here we can see that

$$\#(\mathbb{F}_q^n - \bigcup_{H \in A_q} H) = 4$$

which represent the 4 black points. Also,$\chi_{\mathcal{A}}(3) = 4$, supporting the theorem above.

Now that we've seen an example of the finite field method at work, we will present a proof for this method. This proof constructs two functions f, g and utilizes Möbius Inversion to relate the two functions to the characteristic polynomial.

Proof. Let $x \in L(A_q)$ meaning $\#x = q^{dim(x)}$. Deriving our proof from [S$^+$04],we define functions $f, g : L(A_q) \to \mathbb{Z}$ as

$$f(x) = \#x$$

$$g(x) = \#(x - \bigcup_{y > x} y).$$

With this definition of g, our goal is to show that $g(\mathbb{F}_q^n) = \chi_{\mathcal{A}}(q)$. It can be seen that

$$f(x) = \sum_{y \geq x} g(y)$$

as the right hand side can be thought of as summing over all disjoint sets of points in x which adds up to x itself. Now, using the Möbius Inversion formula, we get

$$g(x) = \sum_{y \geq x} \mu(x, y) f(y) = \sum_{y \geq x} \mu(x, y) q^{dim(y)}.$$

Let $x = \mathbb{F}_q^n$ to get

$$g(\mathbb{F}_q^n) = \#(\mathbb{F}_q^n - \bigcup_{H \in A_q} H) = \sum_{\substack{y \\ y \in L(A_q)}} \mu(y) q^{dim(y)} = \chi_{\mathcal{A}}(q)$$

$\blacksquare$

This formula is extremely powerful as if we can find a way to construct $\#(\mathbb{F}_q^n - \bigcup_{H\in A_q} H)$ for a given q, that construction will represent $\chi_{\mathcal{A}}(q)$ which we can substitute q for -1 to get the number of regions. This method is easily applied in symmetric arrangements such as

$$\mathcal{A}_n = \{x_i - x_j = 0 : 1 \leq i < j \leq n\}$$
$$\mathcal{D}_n = \mathcal{A}_n \cup \{x_i + x_j = 0 : 1 \leq i < j \leq n\}.$$
$$\mathcal{B}_n = \mathcal{D}_n \cup \{x_i = 0 : 1 \leq i \leq n\}$$

We will illustrate using this method to find $r(\mathcal{A}_n)$, $r(\mathcal{B}_n)$, and $r(\mathcal{D}_n)$.

Example ($\mathcal{A}_n$). Our goal is construct $\chi_{\mathcal{A}_n}(q)$. We can first note that given the definition of $\mathcal{A}_n$, $\#(\mathbb{F}_q^n - \cup_{H\in\mathcal{A}_n} H)$ can be thought of as the number of points with coordinates $(x_1, x_2 \dots x_n)$ such that $x_i \neq x_j$ which means all of x_i are distinct elements in $\{0, 1, \dots q-1\}$. This is where having a sufficiently large is important so the elements $\{0, 1, \cdots q-1\}$ are not reused. Then we know that there are q possible values for x_1, $q-1$ possible values for x_2, and so on giving us

$$\chi_{\mathcal{A}_n}(q) = \#(\mathbb{F}_q^n - \bigcup_{H\in A_q} H) = q(q-1)\cdots(q-(n-1)).$$

Hence,

$$r(\mathcal{A}_n) = |\chi_{\mathcal{A}_n}(-1)| = n!.$$

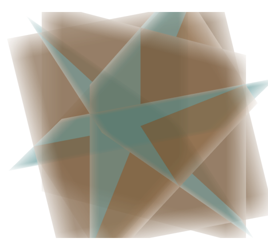
We can look at the example $\mathcal{A}_3$ and verify that it does in fact have $3! = 6$ regions.

This example illustrates how the finite field method can help us transform the problem of counting regions to a simpler combinatorial argument.

Example ($\mathcal{D}_n$). Recall that

$$\mathcal{D}_n = \mathcal{A}_n \cup \{x_i + x_j = 0 : 1 \leq i < j \leq n\}.$$

Below shows $\mathcal{D}_3$.



We can approach this arrangement using a similar approach to what we did for the last problem. However, notice that 0 is a special term this time since when we pick 0, we only restrict off one option for the next choice while picking any other element would restrict off two options. This is because there isn't an element such that $0 + x_i = 0$. Hence, we can split into n cases where the ith case represents picking 0 at the ith position. Counting these, we get the term

$$n \cdot (q-1)(q-3)\cdots(q-(2n-3)).$$

However, we cannot forget the case where we don't pick any zeros.

$$(q-1)(q-3)\cdots(q-(2n-1))$$

Adding these up we get the finished formula

$$\chi_{\mathcal{D}_n}(q) = (q-1)(q-3)\cdots(q-(2n-3))\cdot(q-n+1).$$

$$\chi_{\mathcal{D}_n}(-1) = 2^{n-1}n!$$

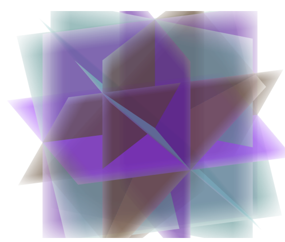
Example ($\mathcal{B}_n$). Recall that

$$\mathcal{B}_n = \mathcal{D}_n \cup \{x_i = 0 : 1 \leq i \leq n\}$$

Using similar reasoning as above, we now want to count the number of points $(x_1, \cdots x_n)$ such that $x_i \neq 0$, $x_i \neq x_j$, and $x_i \neq -x_j$. With 0 not as an option, we have $q-1$ choices for x_1, $q-3$ choices for x_2 since $x_2 \neq \{0, x_1, -x_1\}$ and so on giving us

$$\chi_{B_n}(q) = (q-1)(q-3)\cdots(q-(2n-1)).$$

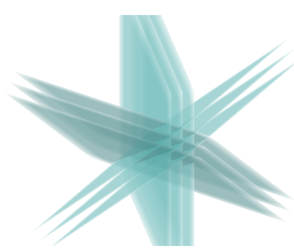
$$r(\mathcal{B}_n) = |\chi_{\mathcal{B}_n}(-1)| = 2^n n!$$



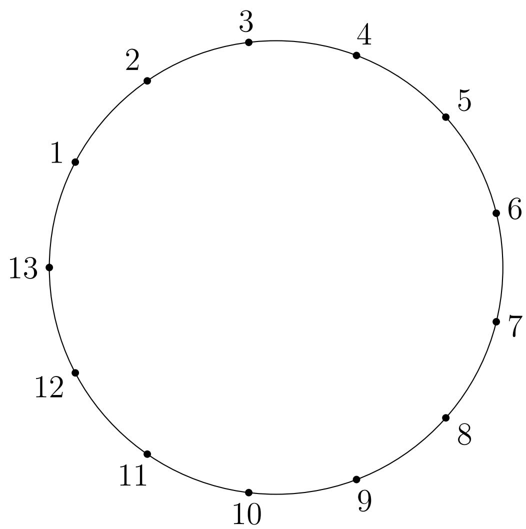
As a side note, $\mathcal{B}_3$ contains all the planes of symmetry for a cube.

Example ($\mathcal{C}_n$). Define $\mathcal{C}_n$, the Catalan arrangement, as

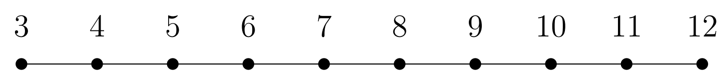
$$\mathcal{C}_n = \{x_i - x_j = -1, 0, 1 \ : \ 1 \leq i < j \leq n\}$$



We will once again find $\chi_{\mathcal{C}_n}$ using the finite field method. We will be using the combinatorial argument stars and bars along the way. We can visualize an equivalent problem as having q points around a circle and we want to count how many ways there are to pick n of those points such that no two points that are chosen are adjacent. This figure shows $q = 13$ as an example



There are obviously q ways to choose the first point on the circle. After we have chosen the first point, we can remove that point and the adjacent two points and stretch the circle into a straight line.



Now we have $q-3$ spots left to choose from for $n-1$ points. Now we can use the stars and bars idea on the number of empty spots and the number of points then multiply the result by $(n-1)!$ since our points are distinguishable. To ensure that each point will not be adjacent to each other, we first take out $n-2$ empty spots that we will add in between the $n-1$ points. Hence, we have $q-3-(n-2) = q-n-1$ total spots left to put our $n-1$ points. From stars and bars, we get the number of ways to do this is

$$\binom{q-n-1}{n-1} = \frac{(q-n-1)(q-n-2)\cdots(q-2n+1)}{(n-1)!}.$$

Now we multiply this by $(n-1)!$ since our points are distinguishable to get

$$\chi_{\mathcal{C}_n}(q) = (q-n-1)(q-n-2)\cdots(q-2n+1)$$

and

$$\chi_{\mathcal{C}_n}(-1) = (-1)(-n-2)(-n-3)\cdots(-2n) = n!\mathcal{C}_n$$

where $\mathcal{C}_n = \frac{1}{n+1}\binom{2n}{n}$ is the nth Catalan number, hence the name Catalan arrangement.

We can see that the finite field method is useful for many classes of arrangements, however, it doesn't work on any arbitrary arrangement. Counting $\#(\mathbb{F}_q^n - \bigcup_{H\in A_q} H)$ for an arbitrary arrangement would be very difficult, hence this is mainly used on symmetric arrangements. This is a major limitation of this method.

7. Further Study

The content of this paper only scratches the surface of the study of hyperplane arrangements. There are many connections between graph theory and how we can translate problems involving hyperplane arrangements to graph theory. These problems can also be generalized into objects such as matroids which provide helpful properties [S$^+$04]. [S$^+$04] contains many open problems in the field in its exercises. As this is an open area of research, many new and interesting results are being published each year.

8. Acknowledgements

The author would like to thank Simon Rubinstein-Salzedo and Andrew Lin for their help in the writing of this paper.

References

[S+04] Richard P Stanley et al. An introduction to hyperplane arrangements. *Geometric combinatorics*, 13(389-496):24, 2004.

[Sam03] Eric Michael Samansky. Arrangements of hyperplanes and zaslavsky's theorem. 2003.