Introduction to Transcendental Number Theory

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What is Transcendence?

Definition (Algebraic Numbers)

A number α is **algebraic** if it is a root of some polynomial $p \in \mathbb{Z}[x]$.

Are there numbers which are not algebraic, which *transcend* the world of algebra?

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What is Transcendence?

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What if we just can't find their polynomials?

Cantor and Countability

Georg Cantor (1845-1918) developed the notion of countability in his first set theory article, in which he provided a straightforward argument to answer our question about the existence of non-algebraic numbers.

Cantor's Argument

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Theorem (Cantor's First Claim)

The algebraic numbers are countable.

Theorem (Cantor's Second Claim)

The real numbers are uncountable. ($|\mathbb{R}| > |\mathbb{N}|$.)

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Corollary (to Cantor's Second Claim)

The complex numbers are uncountable.

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Corollary (to Cantor's Second Claim)

The complex numbers are uncountable.

Since the complex numbers are uncountable and the algebraic numbers are countable, there exist complex numbers which are not algebraic.

Definition (Transcendence)

A number α is **transcendental** if it is not algebraic.

A Fundamental Difficulty

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Almost all numbers are transcendental, but little is known about transcendence. Why?

A transcendental number is defined by *what it is <u>not</u>* rather than *what it is*. We must derive a contradiction from the assumption that it is the solution to any polynomial with integer coefficients. (There are infinitely many such polynomials.)

How to Prove α is Transcendental

1. Assume on the contrary that α is a root of some unspecified polynomial p.
2. Build an integer N using α and the coefficients of p.
3. Find a lower bound A on N.
4. Find an upper bound B on N.
5. Show that N cannot be an integer using A and B.
6. Conclude that since N cannot be an integer, α must be transcendental!

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2. Our integer:

$$N = s!\left(e - \sum_{n=0}^{s} \frac{1}{n!}\right).$$

A Warmup: the Irrationality of e

3. $0 < N$ because

$$N = s!\left(\sum_{n=0}^{\infty}\frac{1}{n!} - \sum_{i=0}^{s}\frac{1}{n!}\right) = \frac{1}{(s+1)} + \frac{1}{(s+1)(s+2)}\cdots. \tag{1}$$

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4. Consider (1). Since $s+1 \geq 2$, $\frac{1}{s+1} < \frac{1}{2}$. By the geometric series formula, we have

$$N = \frac{1}{(s+1)} + \frac{1}{(s+1)(s+2)} < \frac{1}{2} + \frac{1}{2^2} + \cdots = 1.$$

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6. e cannot be written as $\frac{r}{s}$, so it is irrational.

Hermite's Clever Construct

Charles Hermite (1822-1901) was the first to prove that e is transcendental. He did so using an integral derived from the studies of his doctoral student Henri Padé on the best rational approximations of irrational numbers (e^k).
We present Hilbert's modification of the original proof.

Preliminaries

Theorem (Gamma Function)

$$\Gamma(n+1) = \int_0^\infty x^n e^{-x} dx = n!.$$

For any polynomial f with integer exponents

$$\int_0^\infty f(x) e^{-x} dx \in \mathbb{Z}.$$

We choose a special function f, and define

$$S := \int_0^\infty f(x) e^{-x} dx.$$

$$e^k = \frac{e^k S}{S} = \frac{\int_0^\infty f(x) e^{k-x} dx}{S}. \tag{2}$$

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$$e^k = \frac{\delta_k + R_k}{S} := \frac{\int_0^k f(x)e^{k-x}dx + \int_k^\infty f(x)e^{k-x}dx}{S}.$$

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With u substitution ($u = k - x$) and a clever choice of f, we can show that R_k is an integer. We have constructed rational approximations R_k/S of e^k with the same denominator.

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$$N = S \cdot p(e) = S \sum_{k=0}^{d} a_k \frac{R_k + \delta_k}{S} = \sum_{k=0}^{d} a_k (R_k + \delta_k).$$

The Transcendence of e

3. If e is a valid root of p, then $N = S \cdot p(e)$ should be 0. However, with the help of our auxiliary function f, we can show that

$$0 < \sum_{k=0}^{d} a_k R_k,$$

so $0 < |N|$.

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4. Also, we can prove that

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5. Since $0 < |N| < 1$, N is not zero and cannot be an integer.
6. e must be transcendental!

Lindemann and π

Note the following famous relation proved by Euler:

$$e^{i\pi} + 1 = 0.$$

This is a polynomial in e, just with complex exponents instead of integers. Ferdinand von Lindemann (1852 - 1939) noticed this and was able to conquer the transcendence of π with a generalization of Hermite's proof.

Lindemann and π

Theorem (Hermite-Lindemann)

e^α is transcendental for all algebraic nonzero α.

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After modifying Hermite's argument, Lindemann sketched the proof for an even further generalization, which was filled in by Karl Weierstrass and others.

Theorem (Lindemann-Weierstrass)

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be distinct algebraic numbers. Then, for any algebraic nonzero numbers $\beta_1, \beta_2, \ldots, \beta_n$,

$$\beta_1 e^{\alpha_1} + \beta_2 e^{\alpha_2} + \beta_3 e^{\alpha_3} + \cdots + \beta_n e^{\alpha_n} \neq 0.$$

Hilbert's 7th Problem

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Theorem (Hilbert's 7th Problem, Gelfond-Schneider)

Let $\alpha \neq 0, 1$ *be an algebraic number, and* $\beta \neq 0$ *another algebraic number. Then,* α^{β} *is transcendental.*

Hilbert conjectured that this problem would only be solved after the Riemann hypothesis and Fermat's Last Theorem, which thankfully turned out to be false.

The End

Thank you for listening!

Pictures taken from Wikipedia.