Spectral Graph Theory and Ramanujan Graphs

Tudor Braicu
tudor.braicu@menloschool.org

Euler Circle

July 6, 2022

Outline

Outline

Eigenvalues and Eigenvectors

Definition (Eigenvalues and Eigenvectors)

A scalar λ is called an *eigenvalue* of an operator $A : V \to V$ if there exist a *non-zero* vector $\mathbf{v} \in V$ such that

$$A\mathbf{v} = \lambda\mathbf{v}$$

The vector $\mathbf{v}$ is called the *eigenvector* of A (corresponding to the eigenvalue λ).

$$\text{For example: } \mathrm{T} = \begin{pmatrix} -6 & 3 \\ 4 & 5 \end{pmatrix}$$

$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -\frac{1}{3} \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ $\qquad$ $\lambda_1 = -7, \lambda_2 = 6$

Characteristic polynomial

To find the eigenvalues of a matrix M, we have to find the roots of the characteristical polynomial.

Definition

Consider an $n \times n$ matrix M. The characteristic polynomial of M, is the polynomial defined by

$$p_M(\lambda) = det(M - \lambda I)$$

where I denotes the $n \times n$ identity matrix.

Definition (Spectrum)

The spectrum of a matrix is the set of its eigenvalues.

The whole spectrum provides valuable information about a matrix.

Outline

Adjacency Matrix

Definition (Graph)

A graph is a tupel G = (V,E), where V is a set whose elements are called vertices and E is a set of paired vertices, whose elements are called edges.

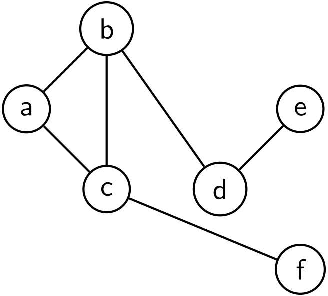
Graphs can be represented in different types of matrices, the most commonly used representation is the following

Definition

Let G be a (finite, undirected) graph with node set $V(G) = 1, \ldots, n$. The *adjacency matrix* of G is defined as the $n \times n$ matrix $A_G = (A_{ij})$ in which

$$A_{ij} = \begin{cases} 1, & \text{if i and j are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

A_G is always symetric.



$$\text{Adjacency matrix } A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Knowing that a graph can be represented as a matrix it raises the question whether the properties of the adjacency matrix can tell us properties of the Graph.

Spectral Graph Theory

Definition

Spectral graph theory is the study of the properties of a graph in relationship to the characteristic polynomial, eigenvalues and eigenvectors of the matrices associated with the graph.

For example one can easily show that for a d-regular graph (each vertex has d edges), for every eigenvalue λ_i of the adjacency matrix $\lambda_i \leq d$.

Properties

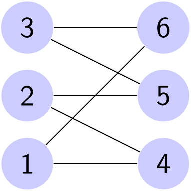
The adjacency matrix of a d-regular graph has in every row (and column) a sum of d.

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

If we take the vector $v = \{1, \ldots 1\}^T$ then d is the eigenvalue and there can't exist one bigger then d.

Properties

If the graph is d-regular and bipartite then we obtain that $\lambda = -d$ is an eigenvalue as well. In fact, all eigenvalues are symetric about 0.



$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Spectral Gap

Let A(G) be the adjacency matrix of a k-regular graph G = (V,E) with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ then we can order them wlog

$$k = \lambda_1 > \lambda_2 \geq \ldots \geq \lambda_{n-1} \geq \lambda_n \geq -k$$

Every k-regular graph has eigenvalues $\lambda_1 = k$ so it's usually referred to as a trivial eigenvalue.

Definition (Spectral Gap)

Given a connected d-regular graph G with adjacency matrix A(G) and associated eigenvalues $d = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, the spectral gap of G is d - λ_2.

Outline

Expanders

Expander graphs are sparse, regular well-connected graphs with many properties. They are quantified using vertex, edge or spectral expansion. Expanders have many applications in computer science including:

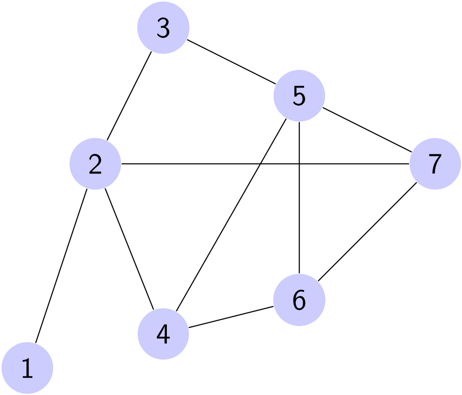
- Error Correcting Codes
- Pseudorandom generators
- Sparse approximation problems
- Major Theorems in Theoretical Computer Science

Definition (Edge boundary)

Let G be a k-regular graph on n vertices and let S be a subset G of vertices of V (G=(V,E)). The *edge boundary* of S denoted by δS is

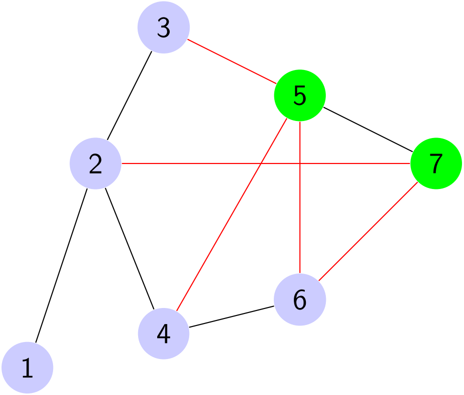
$$\delta S := \{(u, v) \in E : u \in S, v \notin S\} \tag{3.1}$$

Example



Example

Now lets pick: $S = \{5, 7\}$



$$\implies \delta S = \{(4,5),(5,6),(3,5),(2,7),(6,7)\} \implies |\delta S| = 5.$$

Expanders

Since S or the complement of S, has size at most $n/2$ we define the *edge expansion* of G, denoted h(G), as

Definition (Edge Expansion)

The expanding constant of a graph G(V,E) on n vertices is denoted by h(G) where

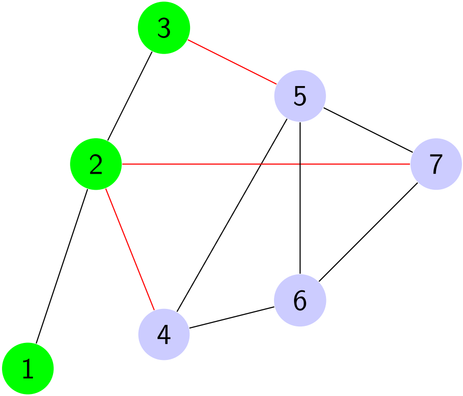
$$h(G) := \min_{S \subset V : |S| \leq n/2} \frac{|\delta S|}{|S|}. \quad (3.2)$$

The bigger h(G) the better the graph connectivity. Therefore, the expanding constant h(G) says how good of an expander a graph is.

Definition

For a fixed $\delta > 0$, we say G is a (k, δ)-expander if $h(G) \geq \delta$.

Example



For $S = \{5, 7\}$ ($|\delta S| = 5$) we get $\frac{|\delta S|}{|S|} = \frac{5}{2}$. However, for $S = \{1, 2, 3\}$ we have $|\delta S| = |\{(2, 4), (2, 7), (3, 5)\}| = 3$, so $\frac{|\delta S|}{|S|} = \frac{3}{3} = 1$. It turns out that the second case is the minimum, thus $h(G) = 1$.

Example

3 important Takeaways:

1. A disconnected graph is not an expander since the expanding constant would be 0. (Pick S to be the unconnected vertex to obtain that)
2. The lowest value of h(G) appeared when we picked the vertex 1, because it only was adjacent to 2. $\implies$ it is more interesting to investigate d-regular graph (they are better expanders)
3. A regular graph with a high degree is very likely to have a good expansion property. A good expander therefore has to have a low degree but a high expanding constant. The challenge is to construct infinite families of fixed degree.

Cheeger Inequality

Theorem (Cheeger Inequality)

Given a connected k-regular graph G=(V,E) with eigenvalues of A(G) $k = \lambda_0 > \lambda_1 \geq \ldots \geq \lambda_{n-1} \geq k$ *then the following inequalities*

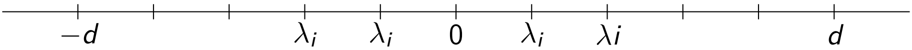
$$\frac{k - \lambda_1}{2} \leq h(G) \leq \sqrt{2k(k - \lambda_1)}. \tag{3.3}$$

are true.

The Cheeger Inequality relates the spectral gap with h(G) which implies that a high spectral gap means a good expander.

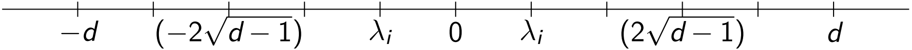
Small eigenvalues

G is a good expander if all non-trivial eigenvalues are small.



Outline

Alon-Boppana Bound



Theorem (Alon-Boppana Bound)

Let G(V,E) be a d-regular graph on n vertices and let A(G) be its adjacency matrix. Let $\lambda_1 > \lambda_2 \geq \ldots \geq \lambda_n$ *be its eigenvalues. Then*

$$\lambda_2 \geq 2\sqrt{d-1}$$

Alon-Boppana (1986): Cannot beat $2\sqrt{d-1}$.

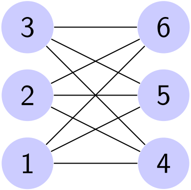
Definition

Ramanujan Graph Let G = (V,E) be a connected d-regular graph with n vertices, and let $d = \lambda_1 > \lambda_2 \geq \ldots \geq \lambda_n \geq -d$ be the eigenvalues of A(G). Define $\lambda(G) = \max_{i \neq 1} |\lambda_i| = max(|\lambda_2|, |\lambda_n|)$. A connected d-regular graph G is a Ramanujan graph if $\lambda(G) \leq 2\sqrt{d-1}$.

- Ramanujan graphs are the best possible expanders.
- Margulius, Lubotzky-Philips-Sarnak (1988): Infinte sequences of Ramanujan graphs exist for d = prime + 1.

Example for a Ramanujan Graph: Bipartite Complete Graph

Adjacency matrix has rank 2, so all non-trivial eigenvalues are 0.



$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Therefore it has the best possible spectral gap, and satisfies the ramanujan property. However it has a high degree k and is not a great expander.

Throughout the years

- Friedman (2008): A random d-regular graph is almost Ramanujan: $2\sqrt{d-1}+\varepsilon$.
- Why are Random Graphs not sufficient?
  - Ramanujan can be constructed more quickly
  - Random graphs are not reliable

Acknowledgements

Thanks to Simon, Nitya and my group consisiting of Logan, Catherine, Ganesh and Ethan. Special thanks to Zipeng Lin for reviewing my paper.

Ask questions! (on discord)