AN INTRODUCTION TO RAMANUJAN GRAPHS

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Abstract. After reading the following paper the reader will have a basic understanding of Ramanujan graphs and their importance. First, we will do some linear algebra background, after which we will focus on the diagonalization of matrices. Before introducing expander graphs, we will give some basic definitions of graph theory. When discussing expander graphs we will explain what a good and bad expander is, and relate it to the graph connectivity. Finally, we will explain the properties of a Ramanujan Graph and give some basic examples and constructions.

1. Introduction

Graph theory enables us to understand complicated processes in the world around us at a single glance. It is now used routinely to model communication networks, and many theorems of graph theory are applied to resolve practical concerns of optimization. There have been attempts to model the human brain using graph theory, with neurons being represented as vertices and synapses being the edges. Though the discipline now has wide application, graph theory as a proper subject of mathematics is relatively of recent origin and its beginning can be traced back to 1736 when Euler solved the famous Königsberg bridge problem using basic ideas of graph theory. Its rapid development is a twentieth century phenomenon. The topic of this paper, Ramanujan graphs as an optimization of expander graphs, connects Linear Algebra with Graph Theory. Ramanujan graphs are named after the famous mathematician Srinivasa Ramanujan and are regular graphs with some extra properties. They can be formally understood in terms of the eigenvalues of the adjacency matrix of a graph:

Definition 1.1. A finite, connected, k-regular graph $G = (V, E)$ with adjacency matrix A(G) is a Ramanujan graph if for all eigenvalues λ, $|\lambda| \neq k$ of A(G) the inequality

$$|\lambda| \leq 2\sqrt{k-1}$$

holds.

Ramanujan graphs have a lot of special properties which are really applicable in the real world as well as in pure math. They are used in computer science, physics, algebraic geometry and number theory. In addition, Ramanujan graphs possess many of the sought after properties of random graphs, which make them optimal for efficient network communication.

In order to be able to fully understand Ramanujan graphs and its importance we will discuss some important Linear Algebra background in 2 and 3. We will introduce Eigenvalues, Eigenvectors and the characteristic polynomial in 2 and talk about why they are interesting for us. In 3 we will talk about the importance of the diagonalization of matrices, and we will proof the spectral theorem, which tells us that any symmetric matrix with real entries is diagonalizable. After that, we will switch over to graph theory and introduce some graph

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theory notion in 4. Then we will talk in 5 about spectral graph theory, which combines graph theory and linear algebra, and we will show what the eigenvalues of the adjaceny matrix can tell about the graph properties. In 6 we will talk about expander graphs and what a well-connected graph means, relating the eigenvalues of a graph to it's connectivity. Finally, in 7 we will conclude this paper with introducing Ramanujan graphs which are the best possible expander graphs.

2. Background to Linear Algebra

In the following section we will learn about the Linear Algebra that is necessary for understanding Ramanujan Graphs. First, we will introduce *Eigenvectors* and *Eigenvalues*, which are important in Linear Algebra as they make Linear Transformations easier. After that we will talk about how to compute the eigenvalues of a matrix.

2.1. **Linear Transformation.**

Linear transformations can be found everywhere in the world, for example a reflection or a rotation is a linear transformation. In the text below we will often call a matrix of a transformation an operator, i.e. a mapping that acts on elements of a space to produce elements of another space.

We define a linear transformation then as:

Definition 2.1. A linear transformation between two vector spaces V and W is a map $T : V \to W$ such that the following hold:

(1) $T(v_1 + v_2) = T(v_1) + T(v_2), \quad \forall v_1, v_2 \in V$, and
(2) $T(\alpha v) = \alpha T(v), \quad \forall \alpha. \in \mathbb{C}$.

Using matrices we can represent information, since the entries in a matrix can represent data. Matrices are used a lot in computer science to model data, but they are also very useful in math, for example we can represent mathematical equations in matrices. Every multiplication by a matrix $M_{m,n}$ can be viewed as a linear transformation T: $R^n \to R^m$, because the matrix multiplication satisfies (1) and (2).

2.2. **Motivation.**

As a rule in math we want to take a general thing and do the best we can to reduce it to a simple thing. In the case of Eigenvalues and Eigenvectors, the general thing is "linear transformations" and the simple thing is "scalar multiplication". Thus we want to reduce linear transformations to scalar multiplication. If we have a linear transformation $A : V \to V$, where A is a $n \times n$ matrix, we can multiply it by itself, take any power of it, or any polynomial. To explain the main idea let us consider the following. Many processes can be described by the equations of the following type

$$x_{n+1} = Ax_n, \quad n = 0, 1, 2, \ldots,$$

where $A : V \to V$ is a linear transformation, and x_n is the state of the system at the time n. Now a common interest is to know the state x_n at the time n, given the initial state x_0. One easily can find the formula $x_n = A^n x_0$. However if n is a very big number then it is very hard to compute the term A^n. Even computers start taking a long time if $n \to \infty$. Fortunately there exists a way around it. Suppose there exists some scalar λ such that $Ax_0 = \lambda x_0$. Then $A^n x_0 = \lambda^n x_0$ so the behaviour of the solution is simplified and can be computed easier and therefore better understood.

2.3. **Eigenvalues and Eigenvectors.**

Definition 2.2. A scalar λ is called an *eigenvalue* of an matrix A if there exist a *non-zero* vector $\mathbf{v} \in V$ such that

$$A\mathbf{v} = \lambda\mathbf{v}.$$

The vector $\mathbf{v}$ is called the *eigenvector* of A (corresponding to the eigenvalue λ).

Note that we write any vector $\mathbf{v}$ as a column vector $\mathbf{v} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. Because the scalar λ just multiplies every row in the matrix representation of $\mathbf{v}$ by λ, one can write instead of $\lambda\mathbf{v} = \lambda I\mathbf{v}$, where I is the Identity-matrix i.e a $n \times n$ square matrix with ones on the main diagonal and 0 elsewhere, we can obtain for the above definition

$$A\mathbf{v} = \lambda\mathbf{v} <=> A\mathbf{v} = \lambda I\mathbf{v} <=> (A - \lambda I)\mathbf{v} = 0$$

Therefore if we know that λ is an eigenvalue, in order to find the eigenvectors we just have to find the nullspace (or kernel) of $A - \lambda I$ (i.e. given a linear map $L : W \to V$, the kernel of L is the vector space of all elements $\mathbf{w}$ of W such that $L(\mathbf{w}) = \mathbf{0}$).

Definition 2.3 (Eigenspace)**.** The nullspace $\mathrm{Ker}(A - \lambda I)$, i.e. the set of all eigenvectors and $\mathbf{0}$ vector, is called *eigenspace*.

The equation $(A - \lambda I)\mathbf{v} = 0$ will always be true if $\mathbf{v}$ is the $\mathbf{0}$ vector, so this won't tell us anything about the transformation. Therefore we are looking for vectors that are not the trivial solution. So we say a scalar λ is an eigenvalue if and only if the equation $(A-\lambda I)\mathbf{v} = 0$ has a non-trivial solution.

Definition 2.4 (Spectrum)**.** The set of all eigenvalues of a matrix A is called *spectrum* of A, and is usually denoted $\sigma(A)$.

The whole spectrum of a matrix provides valuable information about a matrix, which we will see later. In order to find the spectrum of the matrix we have to find it's eigenvalues first. In order to find the eigenvalues one has to be familiar with the determinant. In this paper we will note the determinant of A by det(A). Here are some important properties of the determinant [Tre13]:

(1) The determinant remains unaltered if its rows are changed into columns and the columns into rows.
(2) If all the elements of a row (or column) are zero, then the determinant is zero.
(3) If some row (or column) is a linear combination of the other rows (or column), then the determinant is zero.
(4) The interchange of any two rows (or columns) of the determinant changes its sign.
(5) If all the elements of a row (or column) of a determinant are multiplied by a non-zero constant, then the determinant gets multiplied by the same constant.
(6) Determinant of a Identity matrix is 1.
(7) Let A and B be two matrices, then $\det(AB) = \det(\mathrm{A})\det(\mathrm{B})$

Let A be a square $n \times n$ matrix. Then $A - \lambda I$ has a non-trivial nullspace if and only if it is not invertible. We know that a square matrix is not invertible if and only if its determinant is 0. Therefore λ is an eigenvalue of A if and only if $\det(A - \lambda I) = 0$.

$$\lambda \in \sigma(A), \text{ i.e. } \lambda \text{ is an eigenvalue of A} \iff det(A - \lambda I)$$

Let's compute the eigenvalue of a 2x2 matrix. Note that the determinant of a 2×2 matrix is:

$$det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc. \tag{2.1}$$

Example. Say we have the matrix of transformation $T : V \to V$ with

$$T = \begin{pmatrix} -6 & 3 \\ 4 & 5 \end{pmatrix}.$$

Then in order for $\lambda \in \sigma(T)$, $\det(T - \lambda I) = 0$. So we have

$$\det(A - \lambda I) = 0 \iff \det \begin{pmatrix} -6 - \lambda & 3 \\ 4 & 5 - \lambda \end{pmatrix} = 0$$

By (2.1) we get

$$(-6 - \lambda)(5 - \lambda) - 12 = 0 \iff \lambda^2 + \lambda - 42 = 0$$

Solving this quadratic equation in λ reveals $\lambda_1 = -7$ and $\lambda_2 = 6$. So the spectrum of T is $\sigma(T) = \{-7, 6\}$. Now to find the eigenvectors of the two eigenvalues we have to plug the eigenvalues in the equation $(T - \lambda I)\mathbf{v} = 0$. For $\lambda_1 = -7$ we get

$$\begin{pmatrix} -6 - (-7) & 3 \\ 4 & 5 - (-7) \end{pmatrix} \mathbf{v} = 0 \iff \begin{pmatrix} 1 & 3 \\ 4 & 12 \end{pmatrix} \mathbf{v} = 0$$

The vector $\mathbf{v}$ can be represented as: $\mathbf{v} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. So if multiply the two matrices $\mathbf{v}$ and T we obtain the two equations

$$1x_1 + 3x_2 = 0$$
$$4\text{x}_1 + 12x_2 = 0.$$

We get out of both equations that $x_1 = -3x_2$. So there infinitely many possible eigenvectors but they are linearly dependent. For example $\begin{pmatrix} 1 \\ -\frac{1}{3} \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ are both possible Eigenvectors of the eigenvalue $\lambda_1 = -7$. We can just say that the eigenvectors of λ_1 is any non-zero multiple of $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -\frac{1}{3} \end{pmatrix}$.
With the same procedure we will get that all the eigenvectors of λ_2 is any non-zero multiple of $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$. So we have the eigenspace $\{(1, -\frac{1}{3})^t, (1, 4)^t\}$

Since a 2×2 matrix has a maximum of 2 eigenvalues, we found all the eigenvalues and corresponding eigenvectorx. More generally, if A is an $n \times n$ matrix, the determinant $\det(A - \lambda I)$ is a polynomial of degree n of the variable λ. This polynomial is called the *characteristic polynomial* of A. To find all eigenvalues of A one needs to compute the characteristic polynomial and find all its roots. However, this method of finding the spectrum of an operator is not very practical in higher dimension, because it's hard to find the roots of high polynomials.

2.4. **Characteristic polynomial.**

Definition 2.5 (Characteristical Polynomial)**.** Let A be $n \times n$ matrix. The *characteristic polynomial* of A is the polynomial $p(\lambda)$ given by

$$p(\lambda) = det(A - \lambda I)$$

.

As indicated above, this will give a polynomial of degree n. Based on the fundamental theorem of algebra, this has n not necessarily distinct roots. We will have to find the roots of the polynomial to find the eigenvalues.

The roots (i.e. the eigenvalues) of the characteristic polynomial are not always real. Thus linear transformations have not necessarily real eigenvalues. For example the rotation $R_\alpha, \alpha \neq \pi n$ in $\mathbb{R}^2$ is a real vector space without *real* eigenvalues. Therefore, the complex case (i.e. operator acting in complex vector spaces) seems to be the most natural setting for the spectral theory (i.e. the studies of the eigenvalues and eigenvectors of a matrix). Since $\mathbb{R} \subset \mathbb{C}$, we can always treat a real n x n matrix as an operator in $\mathbb{C}^n$ to allow complex eigenvalues. Treating real matrices as operators in $\mathbb{C}^n$ is typical in the spectral theory, and we will follow this agreement.

If p(z) is a polynomial, and λ is its root (i.e. $p(\lambda) = 0$) then $(z-\lambda)$ divides p(z). The largest positive integer k such that $(z - \lambda)^k$ divides p(z) is called the (algebraic) multiplicity of the root λ. Any polynomial $p(z) = \sum_{k=0}^{n} a_k z^k$ of degree n ($a_n \neq 0$) has exactly n complex roots, counting multiplicity. The words *counting multiplicities* mean that if a root has multiplicity d we have to list (count) it d times. In other words p can be represented as

$$p(z) = a_n(z - \lambda_1)(z - \lambda_2) \dots (z - \lambda_n)$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are its complex roots, counting multiplicities.

There is another notion of multiplicity of an eigenvalue: the dimension of the eigenspace Ker(A - λI) is called *geometric multiplicity* of the eigenvalue λ.

Geometric multiplicity is not as widely used as algebraic multiplicity. So, when people say simply "multiplicity" they usually mean algebraic multiplicity.

Definition 2.6. Two square matrices A and B are said to be similar if there exists an invertible matrix P such that

$$B = P^{-1}AP$$

If two matrices are similar then they have the same rank, trace, determinant and eigenvalues. Moreover, their eigenvalues have the same algebraic and geometric multiplicities. We will prove that two similar matrices have the same characteristic polynomial.

Proof. Note that if $A = PBP^{-1}$ then

$$A - \lambda I = PBP^{-1} - \lambda PIP^{-1} = P(BP^{-1} - \lambda IP^{-1}) = P(B - \lambda I)P^{-1},$$

so the matrices $A-\lambda I$ and $B-\lambda I$ are similar. Therefore from the property (7) of determinant we have

$$\det(A - \lambda I) = \det(B - \lambda I)$$

i.e. characteristic polynomials of similar matrices coincide. ■

3. Diagonalization of matrices

One of the application of the spectral theory is the diagonalization of operators, which means given an operator to find a basis in which the matrix of the operator is diagonal. Such a basis does not always exists, i.e. not all operators can be diagonalized (are diagonalizable). Importance of diagonalizable operators comes from the fact that the powers, and more general function of diagonal matrices are easy to compute. Diagonalizable matrices are especially easy for computations, once their eigenvalues and eigenvectors are known. In this section we will give necessary and sufficient conditions for a matrix to be diagonalizable. At the end of this section we will proof the spectral theorem.

Definition 3.1 (Symmetric matrices)**.** A matrix A with real entries is a symmetric matrix if it is equal to its own transpose.

$$\text{A is symmetric } \iff A = A^t$$

Note that a symmetric matrix is always square.

Definition 3.2 (Diagonal matrix)**.** A diagonal matrix is a matrix in which the entries outside the main diagonal are all 0.

Definition 3.3. Let A be a $n \times n$ matrix. A is diagonalizable if there exists an invertible matrix P such that

$$A = PDP^{-1}$$

where D is a diagonal matrix.

Remark 3.4. A square matrix that is not diagonalizable is called *defective*.

Definition 3.5. A square matrix is called *lower triangular* if all the entries above the main diagonal are zero. Similarly, a square matrix is called *upper triangular* if all the entries below the main diagonal are zero.

Computing eigenvalues (finding roots of the characteristic polynomial) can be quite time consuming, however, there is one particular case, when we can just read them off the matrix. Namely

Proposition 3.6. *The eigenvalues of a triangular matrix (counting multiplicities) are exactly the diagonal entries* $a_{11}, a_{22}, \ldots, a_{nn}$.

By triangular here we mean either upper or lower triangular matrix. We will show a proof for the upper triangular matrix, the proof is similar for the lower triangular matrix. Let's introduce the following lemma for our proof.

Lemma 3.7. *The determinant of an upper triangular (and lower triangular) matrix is the product of it's diagonal entries.*

In order to prove this lemma we will have to introduce some characterization of the determinant first.

Definition 3.8. Let $A = [a_{ij}]$ $i, j = 1, \ldots n$ be a matrix. The **cofactor** of a_{ij} is $A_{ij} = (-1)^{i+j}\Delta_{ij}$, where Δ_{ij} is the determinant of order $n-1$ obtained from A by deleting the i-th row and the j-th column.

The following results are well-known: If $A = [a_{ij}]$, $i, j = 1 \ldots n$ is the square matrix, then

(1) $\det A = \sum_{j=1}^{n} a_{i,j}A_{ij}$ (expansion with respect to the i-th row)
(2) $\det A = \sum_{i=1}^{n} a_{i,j}A_{ij}$ (expansion with respect to the j-th column)

Now let U be an upper triangular matrix:

$$U_n = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots \\ 0 & a_{22} & a_{23} & \dots \\ \vdots & & \ddots & \\ 0 & \dots & 0 & a_{nn} \end{bmatrix}.$$

Now we will proof the lemma (3.7) namely that

$$\det U_n = a_{11}a_{22}\dots a_{nn}$$

by induction.

Proof. For $n = 1$ the assertion is trivial. Let's suppose that for any triangular matrix of order n-1 the determinant is equal to the product of the diagonal entries. From (2), with $i = 1$ we get

$$\det U_n = a_{11}(-1)^{1+1}\Delta_{11} = a_{11}\Delta_{11}.$$

Now Δ_{11} is the determinant of the triangular matrix of order n-1.

$$\begin{bmatrix} a_{22} & a_{23} & a_{24} & \dots \\ 0 & a_{33} & a_{34} & \dots \\ \vdots & & \ddots & \\ 0 & \dots & 0 & a_{nn} \end{bmatrix}$$

By the induction hyptothesis it follows that $\Delta_{11} = a_{22}\dots a_{nn} \implies \det U_n = a_{11}a_{22}\dots a_{nn}$. ■

Now we can prove proposition (3.6)

Proof. If we take a look at $\det(A - \lambda I)$, we have

$$\det\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots \\ 0 & a_{22} & a_{23} & \dots \\ \vdots & & \ddots & \\ 0 & \dots & 0 & a_{nn} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & \dots & \\ 0 & 1 & \dots & \\ \vdots & & \ddots & \\ 0 & \dots & 0 & 1 \end{bmatrix} \right) = \det\left(\begin{bmatrix} (a_{11}-\lambda) & a_{12} & a_{13} & \dots \\ 0 & (a_{22}-\lambda) & a_{23} & \dots \\ \vdots & & \ddots & \\ 0 & \dots & 0 & (a_{nn}-\lambda) \end{bmatrix} \right).$$

Therefore we get

$$\det\left(\begin{bmatrix} (a_{11}-\lambda) & a_{12} & a_{13} & \dots \\ 0 & (a_{22}-\lambda) & a_{23} & \dots \\ \vdots & & \ddots & \\ 0 & \dots & 0 & (a_{nn}-\lambda) \end{bmatrix} \right) = (a_{11}-\lambda)(a_{22}-\lambda)\dots(a_{nn}-\lambda).$$

So the diagonal entries are the roots of the characteristic polynomial. ■

Since a diagonal matrix is a particular case of a triangular matrix (it is both upper and lower triangular) we have the following corollary

Corollary 3.9. *The eigenvalues of a diagonal matrix are its diagonal entries.*

Note that the important following theorems follow from the properties of a diagonal matrix:

Theorem 3.10. *A matrix A admits a representation $A = SDS^{-1}$, where D is a diagonal matrix and S is an invertible one, if and only if there exists a basis of eigenvectors of A.*

Moreover, in this case the diagonal entries of D are the eigenvalues and the columns of S are the corresponding eigenvectors (column number k corresponds to the kth diagonal entry of D).

Theorem 3.11. *If an operator $A : V \to V$ has exactly $n = \dim V$ distinct eigenvalues, then it is diagonalizable.*

Since the eigenvalues of a diagonal matrix $D = \operatorname{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ are exactly $\lambda_1, \lambda_2, \ldots, \lambda_n$, we see that if an operator $A : V \to V$ is diagonalizable, it has exactly $n = \dim V$ eigenvalues (counting multiplicities). Thus we can write the following theorem.

Theorem 3.12. *Let an operator $A : V \to V$ have exactly $n = \dim V$ eigenvalues (counting multiplicities). Then A is diagonalizable if and only if for each eigenvalue λ the dimension of the eigenspace $Ker(A-\lambda I)$ (i.e. the geometric multiplicity of λ) coincides with the algebraic multiplicity of λ.*

Proof. First of all let us note, that for a diagonal matrix the algebraic and geometric multiplicities of eigenvalues coincide, and therefore the same hold for the diagonalizable operators.

Let us now prove the other implication. Let $\lambda_1, \lambda_2, \ldots, \lambda_p$ be eigenvalues of A, and let $E_k := Ker(A - \lambda_k I)$ be the corresponding eigenspaces. Because the system of eigenspaces E_k of an operator A is linearly independent, the subspaces $E_{k,k} = 1, 2, \ldots, p$ are linearly independent.

Let B_k be a basis in E_k. It follows that $B = \cup_k B_k$ is a linearly independent system of vectors.

We know that each B_k consists of dim E_k (= multiplicity of λ_k) vectors. So the number of vectors in B equal to the sum of multiplicities of eigenvalues λ_k. But the sum of multiplicities of the eigenvalues is the number of eigenvalues counting multiplicities, which is exactly n = dim V. So we have a linearly independent system of n = dim V eigenvectors, which means it is a basis. ■

3.1. **Spectral Theorem.** In the above section of Diagonalization we came to the conclusion in theorem (3.12) that a matrix is diagonalizable if and only if for each eigenvalue the dimension of the eigenspace coincides with the algebraic multiplicity of the eigenvalue. The spectral Theorem tells us that every symmetric matrix is diagonalizable. Augustin-Louis Cauchy proved the spectral theorem for symmetric matrices, i.e., that every real, symmetric matrix is diagonalizable. In addition, Cauchy was the first to be systematic about determinants. The spectral theorem as generalized by John von Neumann is today perhaps the most important result of operator theory([FMM+13]). In this section we will look prove the spectral theorem. However, we have to look at some necessary background first.

Lemma 3.13. *Let A be a square matrix $n \times n$ then $det(A) = \Pi_{i=1}^{n} \lambda_i$.*

Recall, that if $x = a + ib$ is a complex number, then $\bar{x} = a - ib$ is the *complex conjugate*. A complex number is real if and only if $x = \bar{x}$. If $M \in \mathbb{C}^{n\times n}$ is a matrix, then M^* denotes the conjugate transpose of M, that is, $(M^*)_{i,j} = \overline{M_{j,i}}$. Notice that if the entries of M are real, then $M^* = M^t$, where M^t is the *transpose* of M.

Definition 3.14 (Hermitian Matrices)**.** We say that a square matrix M is Hermitian if $M = M^*$.

Note that real symmetric matrices are Hermitian. We will use the following inner product on $\mathbb{C}^n$:

Definition 3.15 (Inner product)**.** If **v,w** $\in \mathbb{C}^n$ are two vectors then their inner product (scalar product) is defined as

$$\langle v\ ,\ w\rangle := v * w = \sum_i \overline{v_i} \cdot w_i. \tag{3.1}$$

Notice that, by definition, we have $\langle v\ ,\ w\rangle = (\langle v\ ,\ w\rangle)^*$ and $\langle v\ ,\ v\rangle = ||v||^2$. Note also that, for two matrices A, B, we have $(A \cdot B)^* = B^* \cdot A^*$, and that for every matrix M and every two vectors **x,y**, we obtain:

$$\langle Mx\ ,\ y\rangle = x^* M^* y = \langle x\ ,\ M^* y\rangle. \tag{3.2}$$

Theorem 3.16 (Gram-Schmidt, [Tre13])**.** *Let $v_1, v_2, \ldots, v_n$ be n linear independent vectors each of them with magnitude 1 in a vector Space V containing a scalar product $\langle\ ,\ \rangle$. Then we can always find a orthonormal system $z_1, z_2, \ldots, z_n$.*

Remark 3.17. It is sufficient to construct n orthogonal vectors, because if $w_1, w_2, \ldots, w_n$ are orthogonal ($\langle w_i\ ,\ w_j\rangle = 0,\ i \neq j,\ i, j = 1, \ldots, n$) then $z_i = \frac{w_i}{||w_i||},\ i = 1, \ldots n$ are orthonormal.

Gram and Schmidt constructed inductively the vector z_i: Let $z_1 = x_1$, $z_2 = x_2 - \langle x_2\ ,\ z_1\rangle \cdot z_1$. We note that $\langle z_2\ ,\ z_1\rangle = 0$ using scalar property: $\langle v_1, \alpha v_2 + \beta v_3\rangle = \alpha\langle v_1\ ,\ v_2\rangle + \beta\langle v_1\ ,\ v_3\rangle$. By induction we obtain: $z_k = x_k - \langle x_k\ ,\ z_{k-1}\rangle \cdot z_{k-1} - \langle x_k\ ,\ z_{k-2}\rangle \cdot z_{k-2} \ldots - \langle x_k\ ,\ z_1\rangle z_1$.

Theorem 3.18 (Spectral Theorem, [Tre11])**.** *Let $M \in R^{n\times n}$ be a symmetric matrix with real values entries then there are n real numbers (not necessarily distinct) $\lambda_1, \lambda_2, \ldots, \lambda_n$ and n orthonormal real vectors $x_1, \ldots x_n,\ x_i \in \mathbb{R}^n$ such that x_i is an eigenvector of λ_i.*

Proof. The eigenvalue is a scalar $\lambda \in \mathbb{C}$ for which there exists a vector $x \in \mathbb{R}^n, x \neq 0$ such that

$$Mx = \lambda x. \tag{3.3}$$

The system (3.3) is a homogeneous one. It has a solution $x,\ x \neq 0$ if and only if

$$P(\lambda) = det(M - \lambda I) = 0. \tag{3.4}$$

$P(\lambda)$ is a polynomial of degree n: By the fundamental theorem of algebra there exists $\lambda \in \mathbb{C}$ and $x \in \mathbb{C},\ x \neq 0$ such that (3.4) holds and so (3.3) has a solution $x,\ x \neq 0$.

Lemma 3.19. *If the square matrix M is real and symmetric then its eigenvalues are real.*

This assertion is true for any hermitian matrix. Let us proof this lemma.

Proof. We have

$$\langle Mx\ ,\ y\rangle = \langle x\ ,\ M^*y\rangle = \langle x\ ,\ My\rangle.$$

If x is an eigenvector corresponding to λ, then we get from the properties of the scalar product and from the above equality that

$$\langle Mx\ ,\ x\rangle = \langle \lambda x\ ,\ x\rangle = \bar{\lambda}||x||^2 \text{ and } \langle x\ ,\ Mx\rangle = \langle x\ ,\ \lambda x\rangle = \lambda||x||^2 \tag{3.5}$$

because $||x||^2 \neq 0$, we obtain

$$\bar{\lambda} = \lambda <=> \lambda \in \mathbb{R}. \tag{3.6}$$

■

Let $M \in \mathbb{R}^{n\times n}$ be a real symmetric matrix and $\lambda \in \mathbb{R}$ eigenvalue and a eigenvector $z = x + yi,\ x, y \in R^n$. Using $Mz = \lambda z$, we get $Mx + iMy = \lambda x + i\lambda y$. Note that two complex numbers are equal if and only if the real and imaginary part coincide. Thus we get

$$Mx = \lambda x \text{ and } My = \lambda y \tag{3.7}$$

Because $z \neq 0$ (the eigenvector can't be 0) it follows that x or y is different from 0. Thus M has a real eigenvector.

To finish the proof, we proceed by induction on $n \times n$ (for the dimension of the matrix M): The case $n = 1$ is trivial (the eigenvalue is the entry a_{11} and every vector is an eigenvector, so just pick a real one). Suppose now that the statement is true for the dimension $n-1$. Let λ_1 be a eigenvalue of M and x_1 be a real eigenvector corresponding to λ_1. From $\langle x_1\ ,\ My\rangle = \langle Mx_1\ ,\ y\rangle = \langle \lambda x_1\ ,\ y\rangle = \bar{\lambda}\langle x_1\ ,\ y\rangle$ follows that if $x_1 \perp y$ then $My \perp x_1$.

Let V be the $n-1$ dimensional subspace of $\mathbb{R}^n$ that contains all the vectors orthogonal to x_1. Let $B \in R^{n\times n-1}$ be a matrix that computes a bijective map from R^{n-1} to V (if $b_1, b_2, \ldots, b_n$ is an orthonormal basis for V, then B is the matrix whose columns are the vectors $b_i,\ i = 1, \ldots, n-1$). Let $B' \in \mathbb{R}^{n-1\times n}$ be the matrix such that for every $y \in V$ $BB'y = y$ ($B' = B^t$). We can apply the inductive hypothesis to the matrix

$$M' = B'MB \in \mathbb{R}^{(n-1)\times(n-1)} \tag{3.8}$$

and we find eigenvalues $\lambda_2, \lambda_3, \ldots \lambda_n$ and orthonormal eigenvectors $y_2, y_3, \ldots, y_n$ for M'. For every $i = 2, \ldots, n$ we have

$$B'MBy_i = \lambda_i y_i \tag{3.9}$$

$$\iff BB'MBy_i = \lambda_i By_i. \tag{3.10}$$

Since By_i is orthogonal to x_1 it follows that $MBy_i \perp x_i$ and so $B'BMBy_i = MBy_i$ which implies $MBy_i = \lambda By_i$ and $Mx_i = \lambda x_i$.

Finally we remark that $x_i \perp x_j,\ i \neq j$ because

$$\langle x_i\ ,\ x_j\rangle = \langle By_i\ ,\ By_j\rangle = \langle y_i\ ,\ B^TBy_j\rangle = \langle y_i\ ,\ y_j\rangle = 0 \tag{3.11}$$

Thus we have proven the spectral theorem. ■

Another proof of the spectral theorem can be done using Schur's theorem

Theorem 3.20 (Schur's Theorem)**.** *Let A be an $n \times n$ square matrix with the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ then there exists an unitary matrix $U \in \mathbb{C}_{n\times n}$ such that $U^*AU = T = [t_{(i,j)}]$ where T is upper triangular matrix, having on the main diagonal $t_{(i,i)} = \lambda_i,\ i = 1, 2 \ldots, n$.*

This theorem is of great importance, because the following affirmations result for a square matrix A:
1. det $A = \Pi_{i=1}^{n} \lambda_i$.
2. Tr(A) = $\sum_{i=1}^{n} \lambda_i$.
and many more...

In the following section we will see that we can represent a graph as a matrix with real entries, using the spectral theorem we can then say that the matrix is diagonalizable.

4. Background Graph Theory

In this section we will talk about the notions of graph theory that are necessary to understand spectral graph theory. We will introduce some basic graph notions as well as two ways of how to represent a graph as a martix.

Definition 4.1. A **graph** G = (V_G, E_G) is an ordered pair the vertex set V_G and the edge set $E_G \subseteq V_G \times V_G$ such that
$\forall v \in V_G, (v, v) \notin E$.
$\forall v, w \in V_G, (v, w) \in E_G$ if and only if $(w, v) \in E_G$.

We will refer to V_G simply as V and E_G as E. When discussing edges (elements of the edge set), we will identify (v,w) with (w,v). In other words, we will only consider undirected graphs.

Definition 4.2. Given a vertex $v \in$ V of graph G, the **degree** of v is the number of edges containing to v, i.e.

$$deg(v) := \#(\{(v, w) \in E \,|\, w \in V\}) = \#(\{(w, v) \in E \,|\, w \in V\}). \tag{4.1}$$

We may then determine that $2\#(E) = \sum_{v \in V} deg(V)$.

Corollary 4.3. *G is a k-regular graph, if all vertices in G have degree k.*

Definition 4.4. Given a graph G a **path** (v_i) in G of length n is a sequence of distinct vertices $v_1, \ldots, v_{n+1}$ such that $v_i v_{i+1} \in E$ for each $i \in 1, \ldots, n$. A **cycle** in G of length $n \geq 3$ is a sequence of vertices $v_1, \ldots, v_{n+1}$ such that $(v_1, \ldots, v_n)$ is a path, $v_n v_{n+1} \in E$, and $v_{n+1} = v_1$.

Definition 4.5. A graph is **connected** if for any two vertices v and w, there exists a path in G beginning at v and ending at w.

For the purposes of this paper, we will generally consider connected k-regular graphs G.

Definition 4.6 (Complete bipartite graph)**.** The complete bipartite graph $K_{r,s}$ is a special bipartite graph where every of the first set is connected to every vertex of the second set.

We now have all the basic tools of graph theory and may now proceed to formalize these notions into some algebraic setting.

4.1. Matrix of graphs.

We introduce here representations of a graph in a matrix. The adjacency matrix is the most common used representation of a graph as a matrix.

Definition 4.7. Let G be a (finite, undirected, simple) graph with node set V(G) = $1, \ldots, n$. The *adjacency matrix* of G is defined as the $n \times n$ matrix $A_G = (A_{ij})$ in which

$$A_{ij} = \begin{cases} 1, & \text{if i and j are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

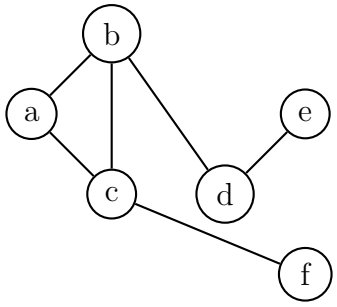
Note that the characteristic polynomial of a graph, is the characteristic polynomial of it's adjacency matrix. There is also another widely used matrix representation which relates to many useful properties of the graph.

Definition 4.8. The *Laplacian* of the Graph is defined as the $n \times n$ matrix $L_G = (L_{i,j})$ in which

$$L_{ij} = \begin{cases} d_i, & \text{if } i = j \\ -A_{i,j}, & \text{if } i \neq j. \end{cases}$$

Here d_i denotes the degree of node i. The Laplacian can be used for example to calculate the number of spanning trees for a given graph.

Example. Lets represent the following graph in the adjacency and Laplacian matrix.



This graph has six vertices a, b, c, d, e and f. When looking at the two matrices, we have to imagine the columns and rows labeled as a, b, c, d, e, f. For example the entry of 1 in row 2 column 3 of the adjacency matrix means that b is connected with c.

$$A_G = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad L_G = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ 0 & -1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix}$$

Observe the following:

(1) The adjacency matrix and the Laplacian are always symmetric, because if b is connected to a, then a is also connected to b.
(2) Both, A_G and L_G have real numbers as entries, thus they are diagonizable (Spectral Theorem)
(3) The sum of the Laplacian in each row and column is 0, while the sum in each row and column of the adjacency matrix is the degree of the vertex.

Knowing that a graph can be represented as a matrix, raises the question whether the properties of the matrix can tell us more about the properties of the graph. This is exactly what Spectral Graph Theory focuses on.

5. Spectral Graph Theory

Spectral Graph Theory is the application of Linear algebra to graph theory and graph algorithms. The application to graph theory comes from associating, in a natural way, a matrix to a graph G = (V,E) and then interpreting the above concepts and algorithms in graph theoretic language.

Definition 5.1. Spectral Graph Theory is the study of the properties of a graph in relationship to the characteristic polynomial, eigenvalues, and eigenvectors of matrices associated with the graph, such as its adjacency matrix or Laplacian matrix.

In this paper we will only work with the adjacency matrix. In the following section we will study some of the properties of the adjacency matrix of a graph. We will introduce some simple connections between the properties of the graph and the properties of the corresponding adjacency matrix.

5.1. **Properties of the spectrum.** The spectrum of a Graph encodes a lot of information about the graph. We will show some simple illustrations of how certain properties of a d-regular graph are reflected in its spectrum.

Definition 5.2. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of a matrix $A \in \mathbb{C}^{n\times n}$. The spectral radius of A is defined as

$$\rho(A) = max\{|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|\}. \tag{5.1}$$

Now lets proof the following theorem:

Theorem 5.3. *Given a graph G with adjacency matrix A(G) and maximum vertex degree $\Delta(G)$, for any eigenvalue λ of A(G), $|\lambda| \leq \Delta(G)$, or in other words $\rho(A) \leq \Delta(G)$.*

Proof. First let λ be an eigenvalue of G. Let us denote with $v = \{c_1, \ldots, c_n\}^T$ the eigenvector corresponding to λ. Let the adjacency matrix of G be A = $[a_{ij}]$. Note that the sum of the i-th row of A is equal to the degree of the i-th vertex of G.

Let $|w_j| = max(|c_1|, |c_2|, \ldots, |c_n|)$. It follows that

$$\sum_{i=1}^{n} a_{ji}w_j = \lambda w_j \implies |\sum_{i=1}^{n} a_{ji}w_j| = |\lambda||w_j| \iff |\sum_{j\in N(i)} w_j| = |\lambda||w_j| \tag{5.2}$$

where N(i) is the set of all i where $a_{ji} = 1$.

From (5.2) we have the following inequality

$$|\lambda||w_j| = \sum_{j\in N(i)} |w_i| \leq \Delta(G)|w_j| \tag{5.3}$$

So we have obtained $|\lambda||w_j| \leq \Delta(G)|w_j|$, because w_j was the biggest component of v it cannot be 0. Thus we have shown that $|\lambda| = \rho(A) \leq \Delta(G)$. ∎

Note that if G is a k-regular graph, then the maximum vertex degree is k, so each of the eigenvalues satisfies $|\lambda| \leq k$. Since a k-regular graph has every vertex with degree k, A(G) has every row/column sum to k. Therefore a k-regular graph, has k always as an eigenvalue of A(G) with the associated eigenvector $v = (1, 1, \ldots)^t$.

A graph G is called *bipartite* if its vertices can be divided into two disjoint sets sets U and V such that no two vertices within the same set are adjacent (connected by an edge). With suitable labelling, we can get that the adjacency matrix of a bipartite graph has the form

$$\begin{pmatrix} 0 & B \\ B^t & 0 \end{pmatrix}$$

where 0 represents a suitable zero matrix and B^t is the transpose of B.

If G is bipartite one can show similarly to above that -k is an eigenvalue of A(G). Moreover, the eigenvalues of the adjacency matrix of a bipartite graph are symmetric about the origin.

Theorem 5.4 (Perron-Forbenius Theorem)**.** *Let A be a positive square matrix. Then*

(1) *$\rho(A)$ is an eigenvalue.*
(2) *$\rho(A)$ has geometric multiplicity 1.*
(3) *$\rho(A)$ has algebraic multiplicity 1.*

From the Perron-Forbenius theorem (see proof [Cai14]) we see that for k-regular graphs $\lambda = k$ is an eigenvalue with multiplicity of 1. Now let's order the eigenvalues of a k-regular graph. Without loss of generality we can order them as

$$k = \lambda_0 > \lambda_1 \geq \ldots \geq \lambda_{n-1} \geq -k. \tag{5.4}$$

We see that for k-regular graphs $\lambda_0 = k$ is always an eigenvalue and because λ_0 only has multiplicity of 1, it is strictly bigger then λ_1. On the other hand, the eigenvalues can't get smaller then $-k$, however, $-k$ is not always an eigenvalue. Only some specific graphs have $-k$ as an eigenvalue for example the bipartite graph. We call the eigenvalue $k = \lambda_0$ the trivial eigenvalue. Since every k-regular graph has k as an eigenvalue, the biggest eigenvalue doesn't tell us anything about the properties of the graph ($-k$ is sometimes also referred to as a trivial eigenvalue). Therefore the second biggest eigenvalue λ_1 is of greater interest because that is where graphs start to differ.

Definition 5.5 (Spectral Gap)**.** Given a connected k-regular graph G with adjacency matrix A(G) and associated eigenvalues $k = \lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_{n-1} \geq -k$, the spectral gap of G is $k - \lambda_1$.

In other words the spectral gap is the degree of every vertex minus its second biggest eigenvalue. Let's compute the eigenvalues and spectrum of a simple graph: the complete graph.

Definition 5.6. A complete graph, denoted K_n is a graph with n vertices in which any two distinct vertices are adjacent. For such a graph the adjacency matrix is given by

$$\mathrm{A} = \begin{bmatrix} 0 & 1 & 1 & \ldots \\ 1 & 0 & 1 & \ldots \\ \vdots & & \ddots & \\ 1 & 1 & \ldots & 0 \end{bmatrix}$$

Example. Let A be the adjacency matrix of a Graph G.

The eigenvalues are the roots of the equation

$$det \begin{bmatrix} -\lambda & 1 & 1 & \ldots \\ 1 & -\lambda & 1 & \ldots \\ \vdots & & \ddots & \\ 1 & 1 & \ldots & -\lambda \end{bmatrix} = 0 \tag{5.5}$$

If we add the rows $2, \ldots n$ to the first row we get

$$\det \begin{bmatrix} n-1-\lambda & n-1-\lambda & n-1-\lambda & \dots \\ 1 & -\lambda & 1 & \dots \\ \vdots & & \ddots & \\ 1 & 1 & \dots & -\lambda \end{bmatrix} = 0$$

Using determinant property (5) from 2 we get

$$<=> (n-1-\lambda) \det \begin{bmatrix} 1 & 1 & 1 & \dots \\ 1 & -\lambda & 1 & \dots \\ \vdots & & \ddots & \\ 1 & 1 & \dots & -\lambda \end{bmatrix} = 0$$

Using determinant properties again and subtracting the first column from every other column we get:

$$<=> (n-1-\lambda) \det \begin{bmatrix} 1 & 0 & 0 & \dots \\ 1 & -\lambda-1 & 0 & \dots \\ \vdots & & \ddots & \\ 1 & 0 & \dots & -\lambda-1 \end{bmatrix} = 0 <=> (n-1-\lambda)(-1)^{n-1}(\lambda+1)^{n-1}$$

So we get that the eigenvalues of the complete graph are $(n-1)$ with multiplicity of 1 and -1 with multiplicity of $n-1$. Since every vertex is connected with every other vertex (except itself) the degree of every vertex is $n-1$. As we see $n-1$ is indeed an eigenvalue with multiplicity of 1. The spectral gap of the complete graph would be $(n-1)-(-1) = n$.

The complete graph actually satisfies the Ramanujan property, which we will come to later. We will see that the spectrum and spectral gap of Ramanujan graphs satisfy several properties that make them particularly interesting and practically relevant objects of study. Before we define what a Ramanujan graph is, however, it is helpful to look at a broader family of spectral graphs that Ramanujan graphs belong to: expander graphs.

6. Expander Graphs

Ramanujan graphs form an important subset of a larger collection called expander graphs. Expander graphs are sparse, regular and highly connected graphs. It is this strange combination of contradictory properties that makes them so important. Expander graphs seem to first have appeared in the 1960's in a fundamental paper of Kolmogorov and Barzdin ([Bar93]) who were studying the network of nerve cells in the human brain. In 1973, Pinsker ([Pin73]) gave the first formal definition. Since then, they have found important applications in the design of optimal communication networks and error correcting codes. It has even been suggested that expander graphs can be applied to study human thought and other problems in neurobiology. So we see that the notion of expansion seems to be woven into the fabric of the cosmos. The most important expanders are k-regular expanders, where $k \geq 3$ is a small constant. Such graphs are not easy to construct. Most of these constructions are based on deep algebra facts. In this section we will look at what a good and bad expander is and connect the expanding properties to the spectral gap.

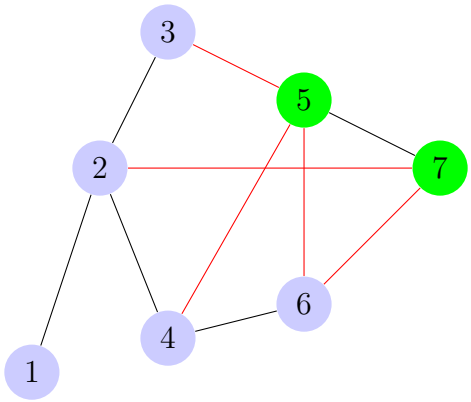
For now we should think of a good expander as a well connected graph. We can quantitatively characterize how good of an expander a graph is by computing the expanding constant (or isoperimetric constant) of a Graph G.

Definition 6.1 ([Tre11])**.** Let G be a k-regular graph on n vertices and let S be a subset G of vertices of V (G=(V,E)). The *edge boundary* of S denoted by δ S is

$$\delta S := \{(u, v) \in E : u \in S, v \notin S\}. \tag{6.1}$$

In other words, the edge boundary of S is the number of edges connecting S to its complement $V - S$. Let's look at an example to illustrate the meaning of the edge boundary.

Example. Lets pick: $S = \{5, 7\}$.



There are 5 edges (highlighted in red) leaving the vertex Set S (the vertices of S are highlighted in green) that are connected to the complement of S.

$$\implies \delta S = \{(4, 5), (5, 6), (3, 5), (2, 7), (6, 7)\} \implies |\delta S| = 5.$$

We see that if there is a high edge boundary for every vertex set S we choose, then that's equivalent with the graph being very well-connected. So the graph has good connectivity if δS is big. Since S or the complement of S, has size at most $n/2$ we define the *edge expansion* (also Cheeger constant, or isoperimetric constant) of X, denoted h(G), as

Definition 6.2 (Edge Expansion, [DSV03])**.** The expanding constant of a graph G(V,E) on n vertices is denoted by h(G) where

$$h(G) := \min_{S \subset V : |S| \leq n/2} \frac{|\delta S|}{|S|}. \tag{6.2}$$

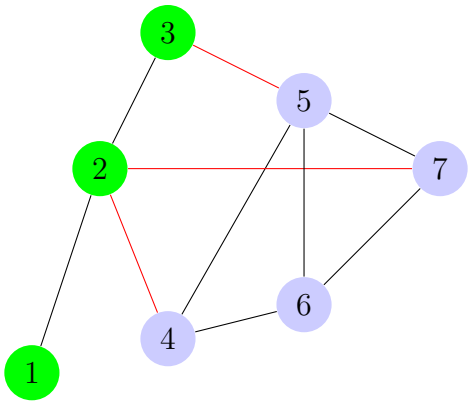
Here, $|A|$ is the cardinal of the set A.

The expanding constant picks the smallest number we get when choosing the set S (in relation to how many vetices S has). So if S has a low edge boundary (is badly connected) then h(G) is small. As in the previous example, we found a good boundary by just picking two well-connected vertices. Therefore in the best case we usually find a good boundary, however, by taking the minimum a high expanding constant means that the whole graph is well-connected, not only parts of it. Therefore the expanding constant h(G) says how good of an expander a graph is.

Definition 6.3. For a fixed $\delta > 0$, we say G is a (k, δ)-expander if $h(X) \geq \delta$.

Definition 6.4 ([DSV03])**.** Consider a family $(G_i)_{i \geq 1}$ of finite connected, k-regular graphs, such that $G_i = (V_i, E_i)$ and $|V_i| \to \infty$ as m $\to \infty$. Then (G_i) is a family of expanders if there exists some $\delta > 0$ so that G_i is a (k, δ)-expander for all i ≥ 1.

Let's calculate the expanding constant of the previous example:



For our previous choice of $S = \{5, 7\}$ ($|\delta S| = 5$) we would get $\frac{|\delta S|}{|S|} = \frac{5}{2}$. However, for $S = \{1, 2, 3\}$ we have $|\delta S| = |\{(2, 4), (2, 7), (3, 5)\}| = 3$, so $\frac{|\delta S|}{|S|} = \frac{3}{3} = 1$. It turns out that the second case is the minimum, thus $h(G) = 1$.

Note the following 3 important takeaways:

(1) A disconnected graph is not an expander since the expanding constant would be 0. (Pick S to be the unconnected vertex to obtain that)
(2) The lowest value of h(G) appeared when we picked the vertex 1, because it only was adjacent to the vertex 2. Therefore d-regular graphs are better expanders since every vertex has the same degree, they avoid this situation.
(3) A regular graph with a high degree is very likely to have a good expansion property. So we don't want it to have a big degree because it's uninteresting to look at such graphs. A lower degree with a strong connectivity is of bigger importance in the application. A good expander, therefore, has to have a low degree but a high expanding constant. The challenge in the application is to construct infinite families of fixed degree, i.e. having a fixed degree k, we want to raise the number of vertices to a big number (ideally to ∞) and still want the good connectivity to remain. That's why these families are called expanders, because we can expand the number of vertices but the good properties remain.

The expanding constant of a graph G is closely tied to the eigenvalues of the adjacency matrix A(G), enabling us to assess the effectiveness of G as an expander in terms of its eigenvalues. We present some bound on the expanding constant h(G). One bound comes from the famous theorem of Dodziuk. It is known as the *Cheeger inequality* and is relevant for constructing explicit expanders. Furthermore it is also used in graph based algorithms in Computer Science.

Theorem 6.5 (Cheeger's Inequality, [DSV03])**.** *Given a connected k-regular graph G=(V,E) with eigenvalues of A(G) $k = \lambda_0 > \lambda_1 \geq \ldots \geq \lambda_{n-1} \geq -k$ then the following inequalities*

$$\frac{k - \lambda_1}{2} \leq h(X) \leq \sqrt{2k(k - \lambda_1)} \tag{6.3}$$

are true.

Proof. We will only proof the first inequality

$$h(G) \geq \frac{1}{2}(k - \lambda_1). \tag{6.4}$$

The basic idea of the proof is the following theorem:

Theorem 6.6 (Courant-Fischer Theorem)**.** *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix with eigenvalues $\lambda_0 > \lambda_1 \geq \ldots \geq \lambda_{n-1}$ then*

$$\min_{w_1,\ldots,w_k\in\mathbb{R}^n} \max_{x\perp w_1,\ldots,w_k} \frac{x^tAx}{||x||^2} = \lambda_k$$

$$\max_{w_1,\ldots,w_{n-k}\in\mathbb{R}^n} \min_{x\perp w_1,\ldots w_{n-k}} \frac{x^tAx}{||x||^2} = \lambda_k.$$

In the proof we will use the results from the above theorem only for λ_1 (in our case $\lambda_0 = k$). Now we can prove inequality (6.4). Let F be a subset of V(G) with $|F| = m$. We define a vector $v = (a_1, \ldots, a_n)^t$ such that

$$v = (n-m)v^F - mv^{VF}$$

where the entries of v^{VF} is 1 in position i if $v_i \in V - F$ and 0 otherwise, and v^F is 1 in position i if $v_i \in V - F$ and 0 otherwise.

Let S be a subset of V, and let us denote by e(S) the number of edges in E(G) that only contain vertices in S. We have

$$a_i = n - m$$

if $v_i \in F$ and $a_i = -m$ if $v_i \in V - F$. With these remarks we obtain

$$v^tv = (n-m)^2m + m^2(n-m) = mn(a-m)$$

$$v^tA(G)v = 2\sum_{(v_i,v_j)\in E} a_ia_j = 2(n-m)^2e(F) - 2m(n-m)|\delta F| + 2m^2e(V-F).$$

So

$$v^tv = mn(n-m) \text{ and } v^tA^Gv = 2(n-m)^2e(F) - 2m(n-m)|\delta F| + 2m^2e(V-F)$$

. Because G is k-regular, the degree of each vertex is equal to k and so $km = 2e(F) + \delta(F)$ Similarly $k(n-m) = 2e(V-F) + |\delta F|$. Therefore we get

$$e(F) = \frac{1}{2}(km - |\delta F|) \text{ and } e(V-F) = \frac{1}{2}(k(n-m) - |\delta F|). \tag{6.5}$$

The last two equalities imply the following equality

$$v^tA_Gv = knm(n-m) - n^2|\delta F|. \tag{6.6}$$

Let $w \in \mathbb{R}^n$ be the vector given by $w = (1, \ldots, 1)^t$. Since $w \cdot v = 0$ from Courant-Fischer for the eigenvalue λ_1 we get

$$\lambda_1 \geq \frac{v^tA(G)v}{||v||^2} = \frac{kmn(n-m) - n^2|\delta F|}{mn(n-m)} = k - \frac{n|\delta F|}{m(n-m)} \tag{6.7}$$

$$\implies k - \lambda_1 \leq \frac{n|\delta F|}{m(n-m)} = \frac{|\delta F|}{|F|}\frac{n}{n-m} \leq 2\frac{|\delta F|}{|F|} \text{ because } (\frac{n}{n-m} \leq 2) \tag{6.8}$$

$$\implies \frac{|\delta F|}{|F|} \geq \frac{k-\lambda_1}{2} \implies h(G) \leq \frac{1}{2}(k-\lambda_1). \tag{6.9}$$

■

This theorem shows that $k - \lambda_1$ (the spectral gap) provides an estimate on the expansion of a graph. In particular, it says that if the spectral gap of G is large then the expanding constant is high, which means G is a good expander. In other words, G is a good expander if all the non-trivial eigenvalues are small. Remember k is fixed and $k - \lambda_1$ has to be big, which means that we want λ_1 to be small.

The following theorem, the Expander mixing lemma, shows that a small second eigenvalue in a graph implies that its edges are "spread out" (i.e. distributed well for good connectivity).

Theorem 6.7 (Expander Mixing Lemma, [HLW06])**.** *Given a connected k-regular graph G = (V,E) on n vertices $v_1, \ldots, v_n$ and disjoint vertex subsets $S, T \subset V(G), S \cap T = \emptyset$. Then the following*

$$||E(S,T)| - \frac{k|S||T|}{n}| \leq \lambda_1 \sqrt{|S||T|}. \tag{6.10}$$

is true.

Let's elaborate here a little more. Let G = (V,E) be a graph and $S, T \subset V(G)$. We denote by E(S,T) the set of edges of G connecting vertices in S to vertices in T. The left-hand side measures the deviation between two quantities: one is $|E(S,T)|$, the number of edges between the two sets; the other is the expected number of edges between S and T in a random graph of edge density k/n, namely $k|S||T|/n$. A small λ_1 (or large spectral gap) implies that this deviation is small, so the graph is nearly random in this sense. The expander mixing lemma gives a bound for how far away from the expected value of E(S,T) a graph can deviate.

Proof. Let $J \in \mathbb{R}^{n \times n}$ be the all 1s matrix. We define $v^{(s)}$ to be the vectors with length n where the ith entry is 1 if $v_i \in S$ and 0 otherwise. We define the vector $v^{(T)}$ similarly. Let $A(G)$ be again the adjacency matrix of G, then we have

$$|E(S,T)| = (v^{(S)})^t A(G)(v^{(T)}) \tag{6.11}$$

$$|S||T| = (v^{(S)})^t \cdot v^{(T)} \tag{6.12}$$

(6.13)

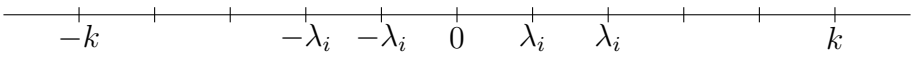
$$\implies \left| |E(S,T)| - \frac{k}{n}|S||T| \right| = \left| v^{(S)t} A(G) v^T - \frac{k}{n}(v^{(S)})^t v^{(T)} \right| = \left| (v^{(S)})^t \left((A(G)) - \frac{k}{n} J \right) v^{(T)} \right|.$$

Using the Cauchy-Schwartz inequality we obtain: $x^T y \leq ||x||||y||$ for any two vectors x and y ,and that $||M|| = \max_{x \neq 0} \frac{||Mx||}{||x||}$ is the largest eigenvalue of M, for any matrix M. In our case, $M = A(G) - \frac{k}{n}J$ with the largest eigenvalue of λ_1. Then, We get that

$$
\begin{aligned}
\left||E(s,T)| - \frac{k}{n}|S||T|\right| &= \left|(v^{(S)})^t\left((A(G)) - \frac{k}{n}J\right)v^{(T)}\right| \\
&\leq ||(v^{(S)})^t||\left|\left|A(G) - \frac{k}{n}J\right|\right||v^T|| \\
&= ||v^{(S)}||\left|\left|A(G) - \frac{k}{n}J\right|\right||v^T|| \\
&= \sqrt{|S|}\left|\left|A(G) - \frac{k}{n}J\right|\right|\sqrt{|T|} \\
&= \lambda_1\sqrt{|S||T|}.
\end{aligned} \tag{6.14}
$$

■

We concluded above that when the spectral gap of G is large then G is a good expander. So it is in our interest right now to determine graphs that are the "best" possible expanders meaning that λ_1 will be as small as possible. We can visualize this in the following graphic:



where we want all the non trivial eigenvalues to be as close to zero as possible.

However, there are bounds on how small the highest non trivial eigenvalue λ_1 can be. We will introduce these in the next section. We will see then that Ramanujan graphs exactly meet the tightest bound which makes them the best possible expander.

7. Ramanujan Graphs

In this section we will finally introduce Ramanujan graphs. First we will look at an example of why the search for expander families, and best possible expanders (Ramanujan graphs) are necessary. Then we will introduce Ramanujan graphs. To conclude this section we will give some insights in how to construct families of Ramanujan graphs.

7.1. Motivation for Ramanujan Graphs.

In computer science or brain research people often search for the fastest path through a Graph. We will show that the fastest path has a connection to λ_1. In order to understand the connections we first have to become familiar with the term of a metric space.

Definition 7.1 ([Tre13])**.** A metric space is an ordered pair (M,d) where M is a set and d is a metric on M, i.e. a function $d : M \times M \to \mathbb{R}$, such that for any $x, y, z \in M$, the following holds:

(1) d(x,y) = 0 <=> x = y
(2) d(x,y) = d(y,x)
(3) d(x,y) $\leq d(x,y) + d(y,z)$
(4) $d(x,y) \geq 0$ for any x,y $\in$ M.

For a finite connected graph G, we can define a metric on G as follows. Given two vertices $u, v \in X$, we let $d(u,v)$ be the length of the shortest path from u to v. The diameter of G is then defined as the maximum value of this distance function. Thus, the diameter is a measure of how fast one can go from one vertex in a graph to another. There is a significant

theorem of Chung ([Chu89]) which gives a bound for the diameter of a k-regular graph in terms of λ_1. A refinement due to van Dam and Haemers is presented next.

Theorem 7.2 ([VDH95]). *If G is a connected k-regular graph with n vertices, the diameter of G is bounded by*

$$1 + \frac{\log 2n}{log(\frac{k+\sqrt{k^2-\lambda_1^2}}{\lambda_1})} \tag{7.1}$$

if G is not bipartite and by

$$2 + \frac{\log 2n}{log(\frac{k+\sqrt{k^2-\lambda_1^2}}{\lambda_1})} \tag{7.2}$$

if G is bipartite.

One should notice that the above shows that to minimize the diameter, we need to minimize λ_1 and this explains our interest in Ramanujan graphs for the application. Now lets prove some bounds for the spectral gap.

7.2. **Bounds for expanders.** Note that a loopless graph has entries of 0 on the diagonal. Now, let G = (V,E) be a k-regular loopless graph with vertices $1, 2, \ldots, n$ and the adjacency matrix A. We note the (i,j)-th entry of A^r as the number of path of length r in the graph. Let v_i be the eigenvector corresponding to $\lambda_i <=> Av_i = \lambda_i v_i$. Then $A^r v_i = A^{r-1}(Av_i) = \lambda_i A^{r-1} v_i = \lambda_i A^{r-2} A v_i = \lambda_i^2 A^{r-2} v_i = \ldots = \lambda_i^r v_i$. So $\lambda_1^r, \lambda_2^r, \ldots, \lambda_n^r$ are the eigenvalues of the matrix A^r. Furthermore, the trace of A^r is

$$tr(A^r) = \lambda_1^r + \lambda_2^r + \ldots + \lambda_n^r \text{(it follows from Schur's theorem for example).} \tag{7.3}$$

Lets prove now the following Theorem:

Theorem 7.3. *Any k-regular, connected graph G on n vertices has some eigenvalue $\lambda_1 \neq k$ such that $|\lambda_1| \geq \sqrt{k}$.*

Proof. First lets consider A^2, then each diagonal entry of A^2 is k. Therefore, $\mathrm{Tr}(A^2) = kn$. On the other hand

$$Tr(A^2) = \lambda_0^2 + \sum_{i=1}^{n-1} \lambda_i^2 = k^2 + \sum_{i=1}^{n-1} \lambda_i^2. \tag{7.4}$$

From the definition of $\lambda(X)$ we get

$$\lambda_i^2 \leq \lambda(G)^2, \; i = 1, \ldots n-1. \tag{7.5}$$

From (7.4) and (7.5) it follows that

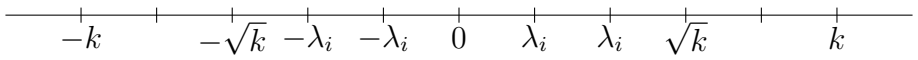
$$Tr(A^2) \leq k^2 + (n-1)\lambda^2(G). \tag{7.6}$$

From plugging in $Tr(A^2) = kn$ in (7.6) we obtain the equation

$$|\lambda(G)| \geq (\frac{n-k}{n-1})^{1/2}\sqrt{k} \geq (\frac{n-k}{n})^{1/2}\sqrt{k} \implies \liminf_{n\to\infty} \lambda(G) \geq \sqrt{k}. \tag{7.7}$$

■

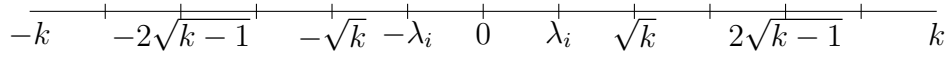
The presented bound above limits how big the spectral gap can be. There has to be one eigenvalue outside of the interval $]-\sqrt{k}, \sqrt{k}[$. In the best case, where $\lambda(G) = \sqrt{k}$ (lowest possible spectral radius), we obtain a spectral gap $k - \lambda_1 = k - \sqrt{k}$ which is more than $\sqrt{k}$ for the degree $k \geq 3$.



So we have a lower bound for the highest eigenvalue meaning that there is a limit for a "good" expander. Noga Alon and Ravi Boppana proved an even higher bound for the eigenvalues, limiting the spectral gap even more. Furthermore, they proved that you can't set a bound higher than this. This is the highest bound that exists to limit the highest non-trvial eigenvalue.

Theorem 7.4 (Alon-Boppana)**.** *Consider a k-regular, connected finite graph G on n vertices with eigenvalues $k = \lambda_0 > \lambda_1 \geq \ldots \geq \lambda_{n-1} \geq -k$ where without loss of generality, λ_1 is the eigenvalue with the largest magnitude not equal to k. Then, $\lambda_1 \geq 2\sqrt{k-1} - \varepsilon$ where $\varepsilon \to 0$ as $n \to \infty$.*

This theorem gives an even tighter bound for the spectral gap: there exists an eigenvalue outside the interval $]-2\sqrt{k-1}, 2\sqrt{k-1}[$



The intuition for the number $2\sqrt{k-1}$ came from the considering the infinite d-regular tree.

A tree is an acyclic (i.e. a graph with no cycles) connected graph. If T is is a tree a vertex of T is called a leaf if it has degree of one. Now we can proof the following theorem

Theorem 7.5. *If T is a tree with maximum degree k, then all the eigenvalues λ of T satisfy*

$$|\lambda| \leq 2\sqrt{k-1} \text{ for } k \geq 2. \tag{7.8}$$

Proof. Let us fix a root r of the tree and let A be the adjacency matrix of T. Then $B = DAD^{-1}$ has the same eigenvalues as A for every invertible diagonal matrix D. Let $\delta > 0$ (to be chosen later) and $D = \text{diag}\{\delta^{l(1)}, \delta^{l(2)}, \ldots, \delta^{l(n)}\}$ where l(i) is the length of the path from r to i. If the entries of the matrix B is $b_{i,j}$ then

$$b_{i,j} = a_{i,j}\delta^{l(i)-l(j)} \tag{7.9}$$

Let λ be a non-zero eigenvalue of B and $x = (x_1, x_2, \ldots, x_n)^t$ an eigenvector corresponding to the eigenvalue λ. We have

$$\lambda x_i = \sum_j a_{i,j}\delta^{l(i)-(l(j))}x_j. \tag{7.10}$$

Let us choose i such that

$$|x_i| = max\{|x_1|, |x_2|, \ldots, |x_n|\}.$$

If i corresponds to the root vertex n then $l(i) = 0$ and every vertex j adjacent to i has length 1.

$$\implies |\lambda_i||x_i| \leq |x_i| \cdot \frac{k}{\delta} \implies |\lambda| \leq \frac{k}{\delta} \tag{7.11}$$

If i is not a root then it has one neighbour occuring in the unique path from the root to i with length $l(i)-1$. Any other neighbor of i (if it is not a leaf) has length $l(i)+1$. As the degree of i is at most k, we have

$$|\lambda x_i| \leq |x_i|(\delta + \frac{k-1}{\delta}) \implies |\lambda| \leq \delta + \frac{k-1}{\delta}. \tag{7.12}$$

Finally, if i is a leaf then it only has neighbors with the length $l(i)-1$. In this case we obtain

$$|\lambda x_i| \leq |x_i|\delta \implies |\lambda| \leq \delta.$$

The middle case suggest that we choose $\delta = \sqrt{k-1}$. To optimize our estimates, and plugging this in in the middle case we ultimately obtain

$$|\lambda| \leq 2\sqrt{k-1}.$$

■

The infinite d-regular tree is an excellent expander, because it can always guarantee the biggest possible spectral gap, as it lies exactly on the bound set by Alon-Bopanna. They are graphs where the eigenvalues exactly meet the bound that we know is impossible to do any better. This motivates us to find more graphs that maximize the spectral gap. Lubotzky-Philips-Sarnak showed that there exist graphs that are exactly on the bound, proving that there are infinetly many of these graphs. Their proof used the Ramanujan conjecture and that's (partially) the reason why we term them Ramanujan graphs.

Definition 7.6 (Ramanujan Graphs)**.** A finite, connected, k-regular graph G = (V,E) with adjacency matrix A(G) is a Ramanujan graph if for all eigenvalues $\lambda \in A_G, |\lambda| \neq k$ we have $|\lambda| \leq 2\sqrt{k-1}$.

Due to achieving the tight bound on λ_1 the expander mixing lemma gives excellent bound on the uniformity of the distribution of the edges in Ramanujan graphs, and any random walks on the graphs has a logarithmic mixing time (in terms of the number of vertices): in other words, the random walk converges to the (uniform) stationary distribution very quickly. Refinements, of the Alon-Bopanna bound have been obtained by Jean-Pierre Serre, who showed the following

Theorem 7.7. *For any $\epsilon > 0$, there is a constant $c = c(\epsilon, k)$ such that for every k-regular graph G on n vertices, the number of eigenvalues λ_i of G such that $\lambda_i > (2-\epsilon)\sqrt{k-1}$ is at least cn.*

Confirming a conjecture of Alon, Friedman showed that many families of random graphs are almost Ramanujan.

Remark 7.8. A random graph is obtained by starting with a set of n vertices and adding successive edges between them at random (the aim of the study in this field is to determine at what stage a particular property of the graph is likely to arise).

Definition 7.9 ([Fri08])**.** Given a random k-regular graph G on k vertices with eigenvalues $k = \lambda_0 > \lambda_1 \geq \ldots \geq \lambda_{n-1}$, then with high probability, $\lambda_1 \leq 2\sqrt{k-1} + \varepsilon$.

It is conjecture that random graphs are Ramanujan with substantial probability (roughly 52%). This raised the question of why the search of families of Ramanujan graphs is still important since we could approximate it with a random graph. It turned out that Ramanujan graphs can be constructed quickly, explicitly and we never fail to generate one.

7.3. **Constructing Ramanujan Graphs.** When constructing Ramanujan graphs it is of great interest to search for the existence of an infinite collection of k-regular Ramanujan graphs for all k. As mentioned in the beginning of this paper, constructing families of Ramanujan graphs is very difficult and uses incredibly deep developments in algebra, algebraic geometry and number theory. That's why we will give here only a brief overview of how to construct Ramanujan graphs.

One way of constructing families of k-regular graphs is using group theory. Several explicit constructions of Ramanujan graphs arise as Cayley graphs and are algebraic in nature. It is shown that a connected Cayley graph is Ramanujan if it satisfies some deep algebraic properties. This enables us to classify certain families of Cayley Graphs as Ramanujan.

Nearly all Ramanujan graphs can be constructed explicitly using results about gasps between consecutive prime numbers. The first constructions of Ramanujan graphs occurred by Lubotzky, Philips and Sarnak, and independently by Margulius. They showed how to construct an infinite family of $(p+1)$-regular Ramanujan graphs, whenever p is a prime number and $p \equiv 1 \pmod 4$. No explicit construction of infinite families of Ramanujan graphs for other degrees is known. A short description of the explicit construction of Ramanujan graphs of degree $p+1$ for every prime p is the following (after M. Ram Murphy).

Example. Let p and q, $p \neq q$, be two prime numbers such that $p, q \equiv 1 \bmod 4$. By number theory the congruence $u^2 = -1 \pmod q$ has an integer solution. By theorems of Lagrange and Jacobi the equation $p = a^2 + b^2 + c^2 + d^2$ has exactly $8(p+1)$ integer solutions. Among these, there are exactly $p+1$ solution were $a \in N^*$ and b, c, d even. To each such solution we associate the matrix

$$\begin{pmatrix} a+ub & c+ud \\ -c+ud & aub \end{pmatrix}$$

which gives $p+1$ matrices in the group $PGL_2(\mathbb{F}_q)$. Let S be the set of these matrices and $G = PGL_2(\mathbb{F}_q)$ the group of cosets invertible 2×2 matrices, where matrices that are the same up to a scalar transformation are considered the same. Then, the Cayley graphs $X(G, S)$ are $(p+1)$-regular Ramanujan graphs, and as q is varied, an infinite family of such graphs can be obtained.

Adam Marcus, Daniel Spielman and Nikhil Srivastava proved the existence of infinitely many k-regular bipartite Ramanujan graphs for any $k \geq 3$. Later they proved that there exist bipartite Ramanujan graphs of every degree and every number of vertices. Michel B. Cohen showed how to construct these graphs in polynomial time. The initial work followed an approach of Bilu and Linial. They considered an operation called a 2-lift that takes a k-regular graph G with n vertices and a sign on each edge, and produces a new k-regular graph G' on 2n vertices. Bilu and Linial conjectured that there always exists a signing so that every new eigenvalue of G' has magnitude at most $2\sqrt{k-1}$. This conjecture guarantees the existence of Ramanujan graphs with degree k and $2^k(k+1)$ vertices for any k-simply start with the complete graph K_{k+1}, and iteratively take 2-lifts that retain the Ramanujan property. Using the method of interlacing polynomials, Marcus, Spielman, and Srivastava ([MSS13]) proved Bilu and Linial's conjecture holds when G is already a bipartite Ramanujan graph, which is enough to conclude the existence result. The sequel proved the stronger statement that a sum of k random bipartite matchings is Ramanujan with non-vanishing probability.

8. Example of Ramanujan Graphs

Lastly, lets look at some examples of Ramanujan graphs. Ramanujan can take any possible form and look very complex. Ramanujan graphs with a lot of vertices and "unpredictabl" structure are of big interest in research. Here, however, we will only discuss some simple, well known graphs.

Recall that earlier we computed the eigenvalues of a complete graph K_n. We had computed for the characteristical polynomial (ch_k)

$$ch_k(\lambda) = (\lambda - n - 1)(\lambda + 1)^{n-1}. \tag{8.1}$$

Let's see if this graph satisfies the Ramanujan Graph property. We have $\lambda(G) = |\lambda_1| = |-1| = 1$ plugging it in the Ramanujan property we get:

$$1 \leq 2\sqrt{(n-1)-1} \leq 2\sqrt{n-2}. \tag{8.2}$$

This inequality is always true for $(n-1) \geq 2$, therefore the complete graph K_n is a Ramanujan graph.

Next we will show that the complete bipartite graph $K_{m,n}$ is Ramanujan as well. We can compute the characteristical polynomial of the complete graph similar to how we did it for the complete graph and we get $ch_K(\lambda) = (\lambda^2 - mn)\lambda^{m+n+2}$. In the d-regular graph $K_{n,n}$ we have $m = n$ so we obtain

$$ch_K(\lambda) = (+(n-1))(\lambda - (n-1))\lambda^{2n+2}. \tag{8.3}$$

Recall that the adjacency matrix of a k-regular bipartite graph has eigenvalues k and -k. Again, a confirmation for our proofs above. Now it is trivial that the complete bipartite graph is Ramanujan as all the non-trivial eigenvalues are zero ($0 \leq 2\sqrt{n-2}$ for $n - 1 \geq 2$).

Note that although the complete bipartite graph and complete graph both satisfy the Ramanujan property, they are bad expanders and not of interest in application, because their degree k is dependent on the number of vertices and thus raises with the number of vertices.

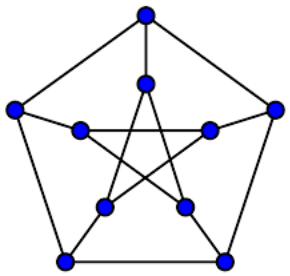


Figure 1. The Petersen graph is a 3-regular graph on 10 vertices and is a Ramanujan graph

As the 3rd example we will show that the well-known Petersen Graph is also a Ramanujan Graph. It's characteristic polynomial is the following:

$$ch_P(\lambda) = (\lambda - 3)(\lambda + 2)^4(\lambda - 1)^5. \tag{8.4}$$

Therefore we have $|\lambda_1| = |-2| = 2$ since $\lambda_0 = k = 3$. This satisfies the Ramanujan property

$$2 \leq 2\sqrt{k-1} = 2\sqrt{k-1} = 2\sqrt{3-1} = 2\sqrt{2}. \tag{8.5}$$

So every non-trivial eigenvalue of G have magnitude less than $2\sqrt{2}$, meaning that the Petersen Graph is a Ramanujan graph.

9. Concluding Remarks

We have presented a brief presentation of Ramanujan Graphs, aimed for the other students who have attended the Euler Circle independent research and paper writing class in the summer of 2022. We have not given a full encyclopedic account of the topic and for this, the reader may look at the surveys such as ([HLW06], [Val97]) as well as the book by Lubotzky ([Lub94]) and the recent book by Kowalski ([Kow19]). I chose this topic because it sounded very interesting and challenging and it connects Linear Algebra (which I studied in my AT math class of 21/22) as well as graph theory which I love. I really enjoyed taking this class and learned a lot about how to become a better mathematician especially in writing a paper and holding talks. Special thanks to Nitya Mani, who helped me answer my questions and provided me with material, and to Simon Rubinstein Salzedo who succesfully organized this great class. Also thanks to Zipeng who reviewed my first draft of this paper and thus contributed to the end result. The theory of Ramanujan graphs is a new world to explore that is fascinating from many perspectives ...

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