Stern-Brocot Tree

Euler Circle

July 10, 2022

Definitions

1. The Mediant
   The Mediant of two fraction $\frac{a}{b}$ and $\frac{c}{d}$ is $\frac{a+c}{b+d}$.
2. Reduced fraction
   A fraction is said to be in its Reduced Form if the fraction $\frac{m}{n}$, where $m, n \in Z$ is expressed in the lowest terms. Therefore m and n have to be coprime.
3. Ancestor
   We say that the fraction $\frac{a}{b}$ is ancestor of fraction $\frac{c}{d}$ if $\frac{a}{b}$ forms the Mediant $\frac{c}{d}$.

How to built Stern-Brocot Tree

1 Start with the 'pseudo-fractions'

$$\frac{0}{1}, \frac{1}{0}$$

2 Insert the mediant

$$\frac{0}{1}, \frac{1}{1}, \frac{1}{0}$$

3 Continue adding all mediants of all neighbouring fractions:

$$\frac{0}{1}, \frac{1}{2}, \frac{1}{1}, \frac{2}{1}, \frac{1}{0}$$

$$\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}, \frac{3}{2}, \frac{2}{1}, \frac{3}{1}, \frac{1}{0}$$

$$\ldots$$

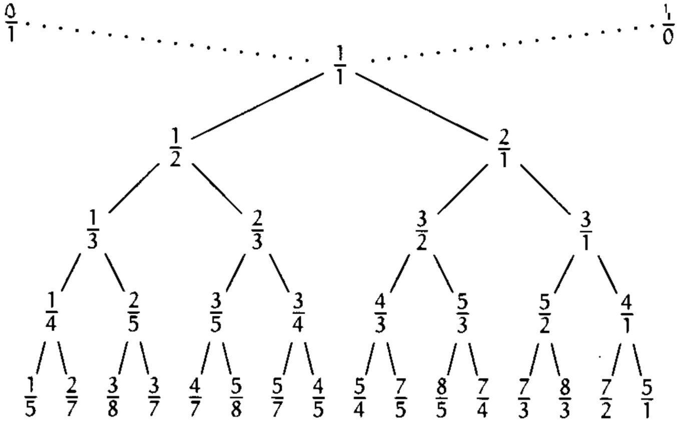

Figure: Stern Brocot Tree

Extended Stern-Brocot Tree

1 Extended Stern-Brocot Tree starts with

$$(0, 1, \frac{1}{0}).$$

2 Every triple $(\frac{m}{n}, \frac{m'}{n'}, \frac{m''}{n''})$ has two child. In every triple, fraction in the middle is Mediant of left and right fraction

$$left\ triple - (\frac{m}{n}, \frac{m+m'}{n+n'}, \frac{m'}{n'}) \qquad right\ triple - (\frac{m'}{n'}, \frac{m'+m''}{n'+n''}, \frac{m''}{n''})$$

Extended Stern-Brocot Tree

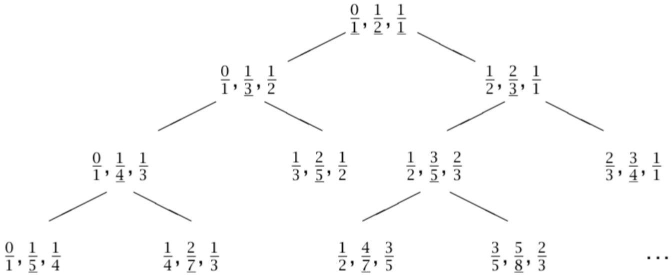

Figure: Extended Stern Brocot Tree

Relation between Stern-Brocot Tree and Extended Stern-Brocot Tree

Lemma

To get Stern-Brocot Tree from the Extended Stern-Brocot Tree, we need to delete left and right fraction in every triple and keep Mediants from the Extended Stern-Brocot Tree.

Proof.

Notice that in the Stern-Brocot Tree $\frac{m}{n}$ and $\frac{m'}{n'}$ are ancestors of fraction $\frac{m+m'}{n+n'}$, for every triple in the Extended Stern-Brocot Tree.
So if we delete right and left fraction in triple, we will leave only Mediants. □

Properties

Lemma

For a, c integer and b, d nonzero integer such that $\frac{a}{b} < \frac{c}{d}$. Then their mediant lies between them, $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.

Property

Lemma

If two fractions $r = \frac{a}{b}$ and $q = \frac{c}{d}$ in reduced form are consecutive, then

$$ad - bc = \pm 1 \tag{1}$$

stern-brocot tree

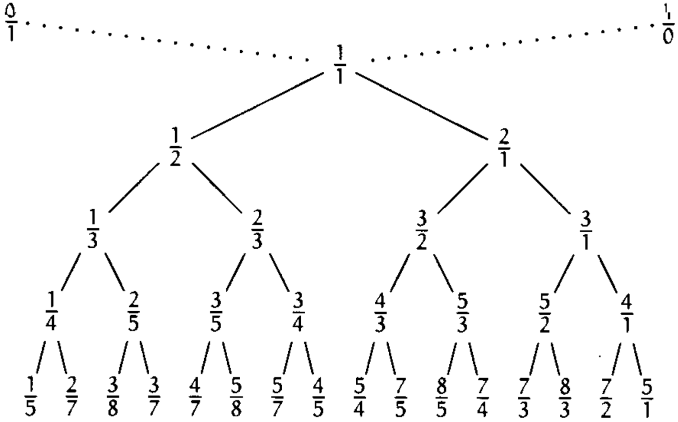

Figure: exmaple

Application

Theorem

For nonzero numbers $a, b \in Z$ *there exist* $x, y \in Z$*, such that* $ax + by = gcd(a, b)$.

Proof.

Consecutive fractions are solutions for Bezout's identity.
And every rational number appears in the tree. □

Dirichlet's Approximation Theorem

Theorem

If α is a positive irrational number, there are infinitely many reduced fractions $\frac{m}{n}$ with $|\alpha - \frac{m}{n}| \leq \frac{1}{n^2}$

proof

Proof.

We will prove that for triple (a, b, c) in the Extended Stern-Brocot Tree corresponding to α, then we say that either a or c is fraction we are looking for.
Every rational number appearsn only finitely many Stern-Brocot triples.
Suppose that $a = \frac{m}{n}$ and $\alpha \in (a, b)$; then $|\alpha - a| \leq |a - b| \leq \frac{1}{n^2}$.
By lemma , for

$$a = \frac{m}{n}, b = \frac{m'}{n'}, c = \frac{m''}{n''}$$

$$mn' - nm' = 1.$$

And n' is at least as large as n.
Similarly we proof for $\alpha \in (b, c)$. □

Hurwiz thorem

Theorem

If α is a positive irrational number, there are infinitely many reduced fractions $\frac{m}{n}$ with $|\alpha - \frac{m}{n}| \leq \frac{1}{\sqrt{5}n^2}$

Binary string

Theorem

Every real positive number a can be uniquely represented as string of L's and R's (maybe empty, maybe infinite).
It helps a lot to computers represent number.

Example

Example

String for The Inverse-Golden Ratio

As we know The Inverse-Golden Ratio equal to $\frac{1}{\phi} = \frac{(-1+\sqrt{5})}{2}$. The sting for The Inverse-Golden Ratio is infinite string LRLRLRLRLRLR ...

Continued fraction and Binary string

Theorem

For every rational number

$$f(R^{a_0}\cdots L^{a_{n-1}}) = a_o + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots + \cfrac{1}{a_{n-1} + \frac{1}{1}}}}}$$

Continued fraction and Binary string

Theorem

For every irrational number $f(R^{a_0}L^{a_1}\cdots) = a_o + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}$

The Inverse-Golden Ratio

Example

$$LRLRLRLR = 0 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\ddots}}}}$$

Thank you for your attention!