PROPERTY AND APPLICATION OF STERN-BROCOT TREE

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Contents

Abstract. Stern-Brocot tree builds by 'childish adding'. Every term is the Tree uses once and all rational numbers are written in their reduced form.

Stern-Brocot Tree has a lot of properties and helps to prove Hurwitz theorem, Dirichlet's theorem, and others.

1. Introduction

In this paper, we will talk about the Stern-Brocot tree and how it is aligned with fundamental mathematical theorems and constants such as Euler's constant, the inverse Golden Ratio, Bezout's identity, Dirichlet's theorem, Hurwitz's theorem, and continued fractions. The Stern-Brocot tree was discovered by Stern in 1858 and Brocot in 1861. The tree is built in a really simple way. It starts from fractions

$$\frac{0}{1}, \frac{1}{0}$$

you just need in every step add numerators and denominator of two fractions with 'childish method'. This repeats infinitely many times, and it brings us a lot of properties. The fundamental one is that every fraction that we received after the 'childish method' is in between its ancestor, so

$$\frac{a}{b} < \frac{(a+c)}{(b+d)} < \frac{c}{d}$$

If you could get one thing from the introduction, then it should be that by inserting new terms with 'childish method', we receive every positive rational number that is symmetrical with regard to reciprocals in each row.

Every term appears exactly once in a reduced form, because of that, each of them has a unique string of L's and R's that show the path from the root of the tree to the fraction we are looking for. For this property we will prove have two different proofs.

This string is beautiful for The Inverse-Golden Ratio, which are

$$LRLRLRLRLRLR\ldots$$

And for Euler Constant is

$$RL^0RLR^2LRL^4RLR^6LRL^8\ldots$$

Extended Stern-Brocot Tree build similar to Stern-Brocot Tree, later we will talk how they connected. We will also show they way how to build it.

Ford Circles use a little bit different way to construction than Stern-Brocot Tree, but fractions corresponding to circles are identical to fractions in Stern-Brocot Tree, we will prove this fact in paper.

It's cool that Stern-Brocot Tree has so many properties, but where they can be used?

Dirichlet's Approximation Theorem is one of the tree theorem where it can be used.

Using properties of the Extended Stern-Brocot Tree and basic algebra, this theorem profs fast.

Hurwitz's theorem which almost like Dirichlet's Approximation Theorem, but harder proofs the same way.

Using properties of the Extended Stern-Brocot Tree and basic algebra,and that ϕ is irrational, we can also proof this.

2. Definitions

Definition 2.1. Child

We say that fraction $\frac{a}{b}$ is child of fraction $\frac{c}{d}$ if $\frac{a}{b}$ appears below the fraction $\frac{c}{d}$.

Example. $\frac{3}{5}$ is child of fraction $\frac{2}{3}$.

Definition 2.2. Left child
We say that fraction $\frac{a}{b}$ is Left child of fraction $\frac{c}{d}$ if $\frac{a}{b}$ appears below to the left than another child fraction.

Example. $\frac{3}{5}$ is left child of fraction $\frac{2}{3}$.

Definition 2.3. Right Child
We say that fraction $\frac{a}{b}$ is Right child of fraction $\frac{c}{d}$ if $\frac{a}{b}$ appears below to the right than another child fraction.

Example. $\frac{3}{4}$ is right child of fraction $\frac{2}{3}$.

Definition 2.4. Ancestor
We say that the fraction $\frac{a}{b}$ is ancestor of fraction $\frac{c}{d}$ if $\frac{a}{b}$ forms the Mediant $\frac{c}{d}$.

Example. $\frac{2}{3}$ is ancestor of fraction $\frac{3}{5}$.

Definition 2.5. Consecutive
Two fractions are consecutive if one of them child and another ancestor.

Example. $\frac{5}{3}$ $\frac{8}{5}$ are consecutive.

Definition 2.6. The Mediant
The Mediant of two fraction $\frac{a}{b}$ and $\frac{c}{d}$ is $\frac{a+c}{b+b}$.

Definition 2.7. Reduced fraction
A fraction is said to be in its Reduced Form if the fraction $\frac{m}{n}$, where $m, n \in Z$ is expressed in the lowest terms. Therefore m and n have to be coprime.

Definition 2.8. n-th level of the Stern-Brocot Tree
The tree has level, to determine fraction's level, we have to find number of all ancestors till the root.

Example. Fractions that located in $3-rd$ level are

$$\frac{1}{3}, \frac{2}{3}, \frac{3}{2}, \frac{3}{1}.$$

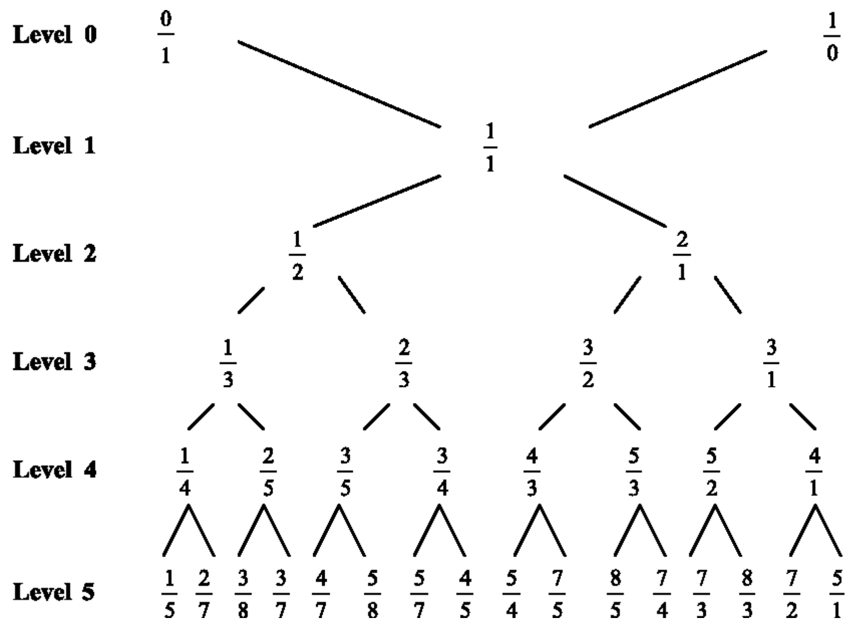

Figure 1. Level in Stern-Brocot Tree

3. How to built the Stern-Brocot tree?

3.1. Stern-Brocot Tree.

1. Start with the 'pseudo-fractions'

$$\frac{0}{1}, \frac{1}{0}$$

2. Insert the mediant

$$\frac{0}{1}, \frac{1}{1}, \frac{1}{0}$$

3. Continue adding all mediants of all neighbouring fractions:

$$\frac{0}{1}, \frac{1}{2}, \frac{1}{1}, \frac{2}{1}, \frac{1}{0}$$

$$\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}, \frac{3}{2}, \frac{2}{1}, \frac{3}{1}, \frac{1}{0}$$

$$\cdots$$

4. Record this numbers as we generate them one by one: below each fraction $\frac{a}{b}$ are the two new fractions introduced to the left and right of $\frac{a}{b}$ at the step directly after $\frac{a}{b}$ is introduced.

As we see, it is quite easy to construct Stern-Brocot Tree but it still have a lot of properties and related to other construction, which we will talk now.

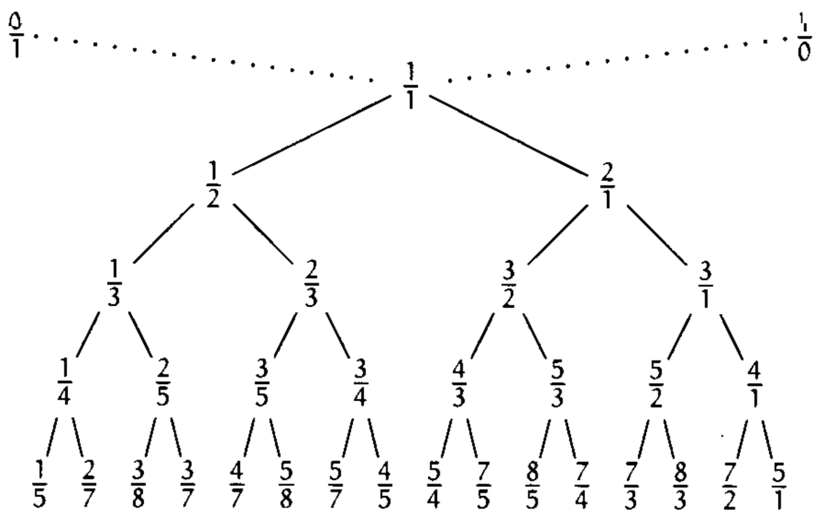

Figure 2. Stern Brocot Tree

3.2. **Extended Stern-Brocot Tree.**

1. Extended Stern-Brocot Tree starts with

$$(0, 1, \frac{1}{0}).$$

2. Every triple $(\frac{m}{n}, \frac{m'}{n'}, \frac{m''}{n''})$ has two child. In every triple, fraction in the middle is Mediant of left and right fraction

$$left\ triple - (\frac{m}{n}, \frac{m+m'}{n+n'}, \frac{m'}{n'}) \qquad right\ triple - (\frac{m'}{n'}, \frac{m'+m''}{n'+n''}, \frac{m''}{n''})$$

3. Example of the left part of the Extended Stern-Brocot Tree:

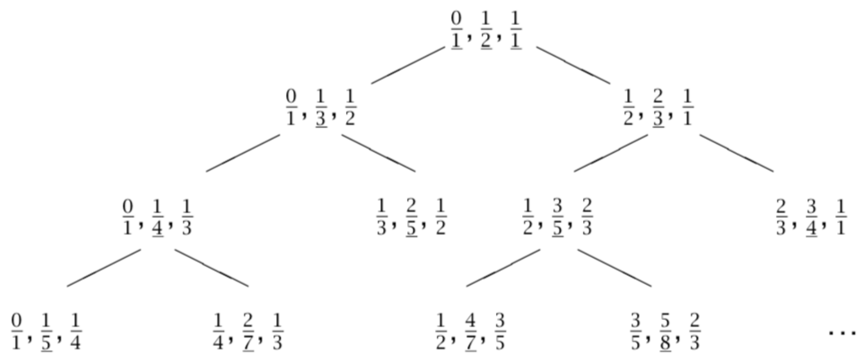

Figure 3. Extended Stern Brocot Tree

Lemma 3.1. *To get Stern-Brocot Tree from the Extended Stern-Brocot Tree, we need to delete left and right fraction in every triple and keep Mediants from the Extended Stern-Brocot Tree.*

Proof. We see that every triple in the Extended Stern-Brocot Tree has form $(\frac{m}{n}, \frac{m+m'}{n+n'}, \frac{m'}{n'})$. But in the Stern-Brocot Tree every rational fraction is used once by theorem 4.4, so we can find fractions $\frac{m}{n}, \frac{m+m'}{n+n'}, \frac{m'}{n'}$ in the Tree.

We notice that $\frac{m+m'}{n+n'}$ is Mediant of left and right fraction by 3.2.

Therefore in the Stern-Brocot Tree $\frac{m}{n}$ and $\frac{m'}{n'}$ are ancestors of fraction $\frac{m+m'}{n+n'}$.

So the left and right fraction in every triple of the Extended Stern-Brocot Tree are ancestors, in Stern-Brocot Tree.

Therefore, from the Extended Stern-Brocot Tree we need to delete all non Mediant fraction, which are left and right fraction in every triple. ■

Even if it seems that Extended Stern-Brocot Tree is useless if we have Stern-Brocot Tree, with Extended Stern-Brocot Tree we can prove Hurwitz's theorem 5.3, Dirichlet's approximation theorem 5.2.

3.3. **Ford circles.** For the pictorial presentation of Ford Circles, we begin with a straight line that can be considered the $x - axis$ in the plane. For any rational number $\frac{a}{b}$, a circle tangent to that point with a diameter $\frac{1}{b^2}$ can be drawn. Some circles are tangent and others are not.

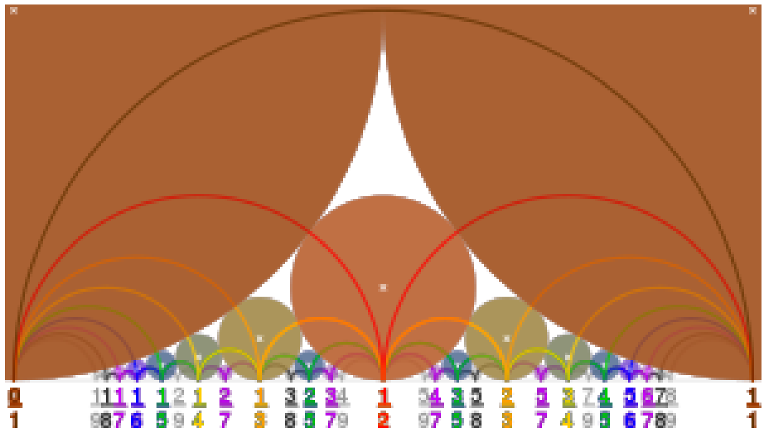

Figure 4. Ford Circles

Theorem 3.2. *The two circles corresponding to* $\frac{a}{b}$ *and* $\frac{c}{d}$ *touch if and only if* $ad - bc = \pm 1$.

Proof. Suppose $b < d$ and $\frac{a}{b} < \frac{c}{d}$, other cases are identical,

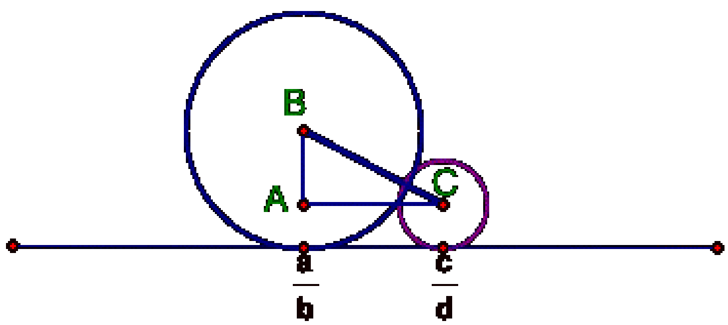

Figure 5. Touching Ford Circles

To use Pythagorean theorem, we need to determine values of AB, BC, CD, so

$$2AB = \frac{1}{b^2} - \frac{1}{d^2}.$$
$$2BC = \frac{1}{b^2} + \frac{1}{d^2}.$$
$$AC = \frac{c}{d} - \frac{a}{b}.$$

Then plugging this values into Pythagoream theorem, we get $bc - ad = 1$.

Therefore two circles are touching each other.

Similarly in reverse, if $bc - ad = 1$ then circles are touching.

Theorem 3.3. *If two circle touch each other then the circle is tangent to them both locates at fraction* $\frac{a+c}{b+d}$.

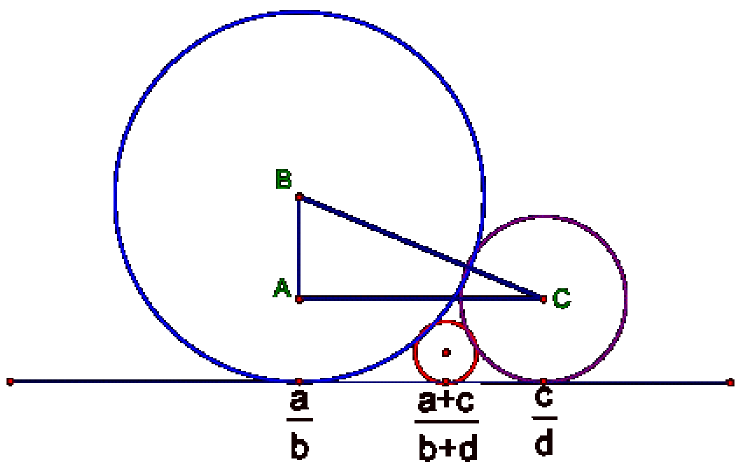

Figure 6. Circle corresponding to the mediant fraction

If we will prove circle, that has diameter $\frac{a+c}{b+d}$, tangent to circle corresponding to $\frac{a}{b}$, analogically that circle tangent to $\frac{c}{d}$, then it is tangent to both of them.

From $bc - ad = 1$, theorem 3.2, we get that $(b+d)c - (a+c)d = 1$ and $(a+c)b - (b+d)a = 1$.

From first equation we get, that $\frac{a}{b}$ and $\frac{a+c}{b+d}$ are touching.

And from the second equation, we get $\frac{c}{d}$ and $\frac{a+c}{b+d}$ are touching.

We proved theorem. ■

Lemma 3.4. *Diameters of all touching circles are form the terms of the Stern-Brocot Tree.*

Proof. We will prove by induction that $n - th$ step of drawing tangent circles to already written, we get terms corresponding to the $n - th$ level of the tree.

For $n = 1$, for $\frac{a}{b} = \frac{0}{1}$ $\frac{c}{d} = \frac{1}{0}$, statement is true.

Suppose for some n is true, then we will proof for $n + 1 - th$ raw.

From the theorem 3.3 we know that, for two circles corresponding to fractions $\frac{a}{b}$, $\frac{c}{d}$, at $n + 1 - th$ step we get a new circle corresponding to fraction $\frac{a+c}{b+d}$.

And in the Stern-Brocot Tree at $n-th$ level for two fractions $\frac{a}{b}$, $\frac{c}{d}$, their mediant at $n+1-th$ raw is $\frac{a+c}{b+d}$.

Therefore, we proved our lemma.

■

4. Properties

In this paragraph we will talk about definitions in the Stern-Brocot tree and properties that tree has.

Lemma 4.1. *For a, c integer and b, d nonzero integer such that $\frac{a}{b} < \frac{c}{d}$. Then their mediant lies between them, $\frac{a}{b} < \frac{a+c}{b+b} < \frac{c}{d}$.*

Proof. We know that $\frac{a}{b} < \frac{c}{d}$, then we have

$$ad < bc.$$

And we know $ad < bc$ mean

$$bad < b^2c,$$

so

$$bad + bcd < b^2c + bcd.$$

However,

$$ad^2 < bcd,$$

then

$$bad + ad^2 < bad + bcd.$$

Therefore,

$$bad + ad^2 < bad + bcd < b^2c + bcd.$$

So,

$$(b+d)ad < (a+c)bd < (b+d)bc,$$

then

$$ad < \frac{bd(a+c)}{(b+d)},$$

and we get

$$\frac{ad}{bd} < \frac{bd(a+c)}{bd(b+d)} < \frac{bc}{bd,}$$

or

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}.$$

■

Corollary 4.2. *The order is respected by horizontal position i.e. larger fractions appear to the right of smaller ones.*

Lemma 4.3. *If two fractions $r = \frac{a}{b}$ and $r = \frac{c}{d}$ in reduced form 2.7 are consecutive, then*

$$ad - bc = \pm 1 \tag{1}$$

Proof. Let's prove by using induction. For $r = \frac{0}{1}$ and $s = \frac{1}{1}$ $0*1-1*1 = 1$.

Suggest that for some r and s it is true, then for their mediant is also true, because

$$a(c + d) - (a + b)c = 1 \tag{2}$$

$$d(a + b) - (c + d)b = 1 \tag{3}$$

are equivalent to our assumption by assumption 1.

For 2: $a(c + d) - (a + b)c = ac + ad - ac - cb = da - ab = 1$

For 3: $d(a + b) - (c + d)b = ad + bd - bc - db = da - ab = 1$

■

Theorem 4.4. *Every rational number $\frac{m}{n}$ in the Stern-Brocot Tree appears in its reduced form and appears exactly once.*

Proof. We will prove in this section, that every fraction are not used more than twice.

Let $\frac{a}{b}$ and $\frac{c}{d}$ be consecutive. Using lemma 4.3 we know that,

$$ad - bc = \pm 1$$

Also if $\frac{a}{b} < \frac{c}{d}$ then

$$\frac{a}{b} < \frac{(a + c)}{(b + d)} < \frac{c}{d}$$

because of lemma 4.1.

Therefore Tree is ordered, so no fraction is used more than twice.

Let $\frac{a}{b}$ be a fraction where $a, b > 0$ and $(a, b) = 1$.

In this section we will prove that every fraction will appear in the tree. Until, we will not meet fraction, we are looking for, then we have

$$\frac{m_1}{n_1} < \frac{a}{b} < \frac{m_2}{n_2}$$

where $\frac{m_1}{n_1}$ and $\frac{m_2}{n_2}$ are consecutive.

It easy to see that

$$m_1 b < n_1 a \text{ and } n_2 a < m_2 b,$$
$$n_1 a - m_1 b > 0 \text{ and } m_2 b - n_2 a > 0,$$
$$n_1 a - m_1 b \geq 1 \text{ and } m_2 b - n_2 a \geq 1.$$

because of that,

$$(m_2+n_2)(n_1a-m_1b)+(m_1+n_1)(m_2b-n_2a) \geq m_1+n_1+m_2+n_2.$$

Simplifying, we get

$$m_2n_1a-n_2m_1b-m_2m_1a+n_1m_2b \geq m_1+n_1+m_2+n_2.$$
$$n_1m_2(a+b)-m_1n_2(a+b) \geq m_1+n_1+m_2+n_2.$$
$$(a+b)(n_1m_2-m_1n_2) \geq m_1+n_1+m_2+n_2$$
$$a+b \geq m_1+n_1+m_2+n_2$$

This shows that there would be at most $(a+b)$ steps of mediants before $\frac{a}{b}$ appears in the Stern-Brocot Tree.

■

This properties properties are essential to proof theorems and other lemmas, so even if they look simple they are very usable.

Lemma 4.5. *Each row of the tree is symmetrical with regard to reciprocals.*

$$\frac{1}{4}, \frac{2}{5}, \frac{3}{5}, \frac{3}{4}, \frac{4}{3}, \frac{5}{3}, \frac{5}{2}, \frac{4}{1}$$

5. Application

5.1. **Bezout's identity.**

Theorem 5.1. *For nonzero numbers $a, b \in Z$ there exist $x, y \in Z$, such that $ax+by = \gcd(a,b)$.*

Proof. Let's simplify and delete both sides to $\gcd(a,b)$, we get that for nonzero coprime numbers $a, b \in Z$ there exist $x, y \in Z$, such that $ax+by=1$.

Using 2.7 we know that every fraction $\frac{a}{b}$ is reduced and by theorem 4.4 we know that every fraction is used once, so we can now proof that for every fraction in the Stern-Brocot Tree exist $x, y \in Z$, such that $ax+by=1$.

Using property that for every fraction $\frac{a}{b}$ by lemma 4.3 its consecutive fraction $\frac{c}{d}$, such that $ad-bc=1$, where $a,b,c,d \in N$.

Since, we choose x, y, so we can make a sum or difference, just changing sign before x or y.

Summarizing, in Stern-Brocot Tree we can find any pair of a, b coprime, and for any fraction $\frac{a}{b}$, its consecutive fraction in module are x and y we are searching. ■

In this theorem were useful properties 4.4, and 4.3 and how we defined what consecutive fraction in 2.5.

5.2. **Dirichlet's Approximation Theorem.**

Theorem 5.2. *If α is a positive irrational number, there are infinitely many reduced fractions $\frac{m}{n}$ with $|\alpha - \frac{m}{n}| \leq \frac{1}{n^2}$.*

Proof. We will prove that for triple (a, b, c) in the Extended Stern-Brocot Tree corresponding to α, then we say that either a or c is fraction we are looking for.

Every rational number appears in only finitely many Stern-Brocot triples.

Suppose that $a = \frac{m}{n}$ and $\alpha \in (a, b)$; then $|\alpha - a| \leq |a - b| \leq \frac{1}{n^2}$.

By lemma 4.3, for

$$a = \frac{m}{n}, b = \frac{m'}{n'}, c = \frac{m''}{n''}$$

$$mn' - nm' = 1.$$

And n' is at least as large as n.

Similarly we proof for $\alpha \in (b, c)$. ∎

Nest theorem, looks similar to Dirichlet's Approximation Theorem, but more general case, of what we proved before.

5.3. **Hurwitz theorem.**

Theorem 5.3. *If α is a positive irrational number, there are infinitely many reduced fractions $\frac{m}{n}$ with $|\alpha - \frac{m}{n}| \leq \frac{1}{\sqrt{5}n^2}$.*

Proof. We want prove that in any triple (a, b, c) corresponding to α in the Extended Stern-Brocot Tree, either a, b, c, satisfy.

We will show the case if $\alpha > b$, because if $\alpha < b$, we replace α by $1 - \alpha$, a by $1 - c$, c by $1 - a$, and this case become identical to first case.

In the Extended Stern-Brocot Tree any triple (a, b, c) corresponding to α, to Hurwitz conclusion must satisfy either a, b, c. We will prove for case $\alpha > b$, if $\alpha < b$ we can replace $\alpha = 1 - \alpha, a = 1 - c, c = 1 - a$.

Then we can say, that $\alpha \in (a, b)$. Suppose that a, b, c do not satisfy to the theorem and we will prove that our assumption is wrong.

$$a = \frac{m}{n}, b = \frac{m'}{n'}, c = \frac{m''}{n''}$$

then

$$\alpha - \frac{m}{n} \geq \frac{1}{\sqrt{5}} n^2 \tag{1}$$

$$\alpha - \frac{m'}{n'} \geq \frac{1}{\sqrt{5}} (n')^2 \tag{2}$$

$$\alpha - \frac{m''}{n''} \geq \frac{1}{\sqrt{5}} (n'')^2 \tag{3}$$

Using lemma 4.3, and adding 1 and 2, we get

$$\sqrt{5} n n' \leq n^2 + (n')^2.$$

Doing similarly, adding 2 and 3, we have

$$\sqrt{5} n' n'' \leq (n')^2 + (n'')^2.$$

Adding two inequalities that we received , we get,

$$\sqrt{5} n'(n + n'') \leq n^2 + 2(n')^2 + (n'')^2.$$

Using the definition of mediant 2.6, we know that $n'' = n + n'$, simplifying, it is equal to

$$2(n - \frac{n'}{\phi})^2 \leq 0,$$

where ϕ is the golden mean.

It is possible only if $n = \frac{n'}{\phi}$, as we know ϕ is irrational, which is impossible.

■

As we noticed knowing properties of some constant, and unique numbers in mathematics can be used in the Stern-Brocot Tree, and later we also will use them.

6. Binary string

Definition 6.1. Moves to left and to right in the algorithm

To find path we move to Left or Right branch. We move left, if the left triple (m, n, k) such that $a \in (m, n)$, analogously for right move, where a is real positive number for which we want to find string.

When we build Extended Stern-Brocot tree, every triple (m, n, k) such that n is the Mediant of m, k by lemma 3.1.

By lemma 4.1 we know, that Mediant of two fractions locates in between. Hence we move left or right until a is equal to Mediant of two fractions or infinitely many times if α is rational.

Theorem 6.2. *Every real positive number a can be written as string of L's and R's (maybe empty, maybe infinite).*

Theorem 6.3. *Every rational number a has unique representation of L's and R's, when we are looking for a, we look only such a that placed at the Mediant of every triple.*

Proof. By theorem 4.4 every fraction appears exactly once, then in the tree do not exist equal fraction in the Stern-Brocot Tree, what means there do not exist equal fraction in the Mediant place in every triple of the Extended Stern-Brocot Tree. Therefore every rational positive number a has no more than one path.

Let's prove that different rational numbers a, b do not have same string.

We will prove by induction.

We will do induction by n, where n is the length of the string, since string of a, b are equal, then their length are also equal.

For $n = 1$, it means that from Mediant of the first triple of the Extended Stern-Brocot Tree we moved to left or to right, and a, b are ended their path at the same fraction. What means they are equal.

Suppose for n the statement is true.

Then for $n+1$, we now that at $n's$ step they are equal, from $n's$ letter they both move or left or right. Where they again are equivalent. ■

Theorem 6.4. *Every irrational number α has a unique representation of L's and R's, when we are looking for a, we look only such a that placed at the Mediant of every triple.*

Proof. By theorem 6.2, we know that any irrational number has a string.

Let's suppose that for some irrational number α exist two different strings.

Then we know this both strings at some point will have different letter, suppose that first string has at $n - th$ place L and the second letter R at $n - th$ place.

We know that tree is order by corollary 4.2. Therefore, fraction corresponding to first string anyway is going to be smaller than the second string.

So this strings correspond to different numbers.

■

Definition 6.5. Fraction corresponding to string S, we define as

$$f(S)$$

Theorem 6.6. *If we have string of L's and R's, what fraction has this string?*

Proof. If $\frac{m}{n}$ and $\frac{m'}{n'}$ are the closest fraction to preceding and following terms in the tree, for the string $f(S)$. Then $f(S) = \frac{m+m'}{n+n'}$.

Let's look at a $2 * 2$ matrix

$$A(S) = \begin{pmatrix} n & n' \\ m & m' \end{pmatrix}$$

Is we move from this position to the left, then

$$A(SL) = \begin{pmatrix} n & n+n' \\ m & m+m' \end{pmatrix}$$

equavalent to

$$A(SL) = \begin{pmatrix} n & n' \\ m & m' \end{pmatrix} * \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = A(S) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Simlarly for Right move,

$$A(SR) = \begin{pmatrix} n+n' & n' \\ m+m' & m' \end{pmatrix} = A(S) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

So, eft and Right $2 * 2$ matrix are

$$L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

So we got formula by induction. Therefore this construction gives answer for the question, and each fraction has unique string. ∎

Corollary 6.7. *Each node in that tree can be represented as a sequence of L's and R's, say*

$$R^{a_0} L^{a_1} R^{a_2} L^{a_4} \cdots R^{a_{n-2}} L^{a_{n-1}},$$

where $a_0, a_1, a_2 \cdots, a_{n-2} \geq 1$ *and* $a_{n-1} \geq 0$.

Definition 6.8. The fraction $\frac{1}{1}$ corresponds to the empty string.

Definition 6.9. Irrational numbers

Irrational numbers do not appear in Stern-Brocot tree, but all the rational numbers that are "close" to irrational number to them do. Therefore string is infinite for irrational numbers.

Example. String for The Inverse-Golden Ratio

As we know The Inverse-Golden Ratio equal to $\frac{1}{\phi} = \frac{(-1+\sqrt{5})}{2}$. The sting for The Inverse-Golden Ratio is infinite string LRLRLRLRLRLR . . .

Example. String for Euler's constant.

The string for Euler's Constant is infinite string $RL^0RLR^2LRL^4RLR^6LRL^8\ldots$

RRLRRLRLLLL R L R R R R R R

$\frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \frac{5}{2}, \frac{8}{3}, \frac{11}{4}, \frac{19}{7}, \frac{30}{11}, \frac{49}{18}, \frac{68}{25}, \frac{87}{32},$ -------- $\frac{106}{39}, \frac{193}{71}, \frac{299}{110}, \frac{492}{181}, \frac{685}{252}, \frac{878}{323}, \frac{1071}{394}, \frac{1264}{465}$""

Definition 6.10. Semiconvergents

The mediants of these triples, that built path from the root to α we are looking for, are called the semiconvergents of α.

Example. For $\alpha = \frac{5}{7}$, triples are

$$(0, 1, \infty), (1, \frac{1}{2}, 1), (\frac{1}{2}, \frac{2}{3}, 1), (\frac{2}{3}, \frac{3}{4}, 1), (\frac{2}{3}, \frac{5}{7}, \frac{3}{4}).$$

Semiconvergents

$$1, \frac{1}{2}, \frac{3}{4}, \frac{5}{7}.$$

Example. For $\alpha = 1/\phi$

$$1, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \ldots$$

We see Fibonacci's sequence.

7. Acknowledgements

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References

[1] Sinje Bulter Robin Kirkham. Ford's touching circles.
[2] Cut the knot. The stern-brocot tree.
[3] Charles Alba and Nathan Roy. The farey sequence, stern brocot tree and euclids theorem. *arXiv preprint arXiv:2011.06318*, 2020.
[4] Gary D Knott. The stern-brocot tree. *Civilized Software Inc*, 12109, 2019.
[5] Matt Baker. The stern-brocot tree, hurwitz's theorem, and the markoff uniqueness conjecture.
[6] Ronald L Graham, Donald E Knuth, Oren Patashnik, and Stanley Liu. Concrete mathematics: a foundation for computer science. *Computers in Physics*, 3(5):106–107, 1989.