# Quantum Error Correction

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# Significance

Quantum computers are powerful!
But are also very error-prone.

# Classical Computing

The standard unit of information is the bit, which can take the value 0 or 1. We can combine 0's and 1's to make letters, numbers, and words.
We have classical gates that act on these 0's and 1's.

$$1 \oplus 0 = 1, 0 \oplus 0 = 0, 1 \oplus 1 = 0$$

$$0 \; AND \; 0 = 0, 1 \; AND \; 0 = 0, 1 \; AND \; 1 = 1$$

# Classical Errors

Classical computing is also prone to errors!

1. Poorly calibrated gates:

$$1 \oplus 0 = 0$$

2. External noise and interference

# Bit-Flip Channel

Bit-flip channel: If we send a bit(0 or 1) through the channel, the output is flipped with probability $p$, and remains the same with probability 1-$p$.

# Classical Error Correction

We can protect states from bit-flip errors with repetition.

$$0 \mapsto 000, 1 \mapsto 111$$

Now, if we want to send the 0 state, we instead send the state 000. If a bit-flip occurs on the second bit:

$$000 \mapsto 010$$

We can simply look at what state the majority of our bits are in, and take that value.

# Dirac Notation

A ket $|x\rangle$ is a way of representing a column vector:

$$|x\rangle = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

# Quantum Computing

Quantum computing uses qubits, not bits. The quantum equivalent of the 0 and 1 states are the vectors:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

States can also exist in *superpositions* of the $|0\rangle$ and $|1\rangle$ states:

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

# Measurement

Measurement is an important part of quantum computing. When we measure a state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, we expect the following probabilities:

$$P(|0\rangle) = ||\alpha||^2$$

$$P(|1\rangle) = ||\beta||^2$$

And the state "collapses" to either $|1\rangle$ or $|0\rangle$, depending on the result of the measurement.

# Operators

General operator U:

$$U |\psi\rangle = |\psi\rangle'$$

An operator acts on a quantum state to transform it to another state. Single-qubit operators can be represented as 2x2 matrices. Commuting operators:

$$AB |\psi\rangle = BA |\psi\rangle$$

Anti-Commuting operators:

$$CD |\psi\rangle = -DC |\psi\rangle$$

# Named Operators

Some important operators (Pauli group $\mathcal{P}$):

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \; Y = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

Act on the basis states:

$$X\left|0\right\rangle = \left|1\right\rangle, X\left|1\right\rangle = \left|0\right\rangle$$

$$Z\left|0\right\rangle = \left|0\right\rangle, Z\left|1\right\rangle = -\left|1\right\rangle$$

# Quantum Errors

There are three big challenges to QEC:

1. No-Cloning Theorem:
   There is no operator $U_c$ that can perform the following mapping:
   $$U_c(|\psi\rangle\,|\phi\rangle) \mapsto (|\psi\rangle\,|\psi\rangle)$$
2. Continuous errors
3. Measurement collapses state information.

How can we overcome these challenges?

# Encoding

The first part of quantum error correction is the encoding process. In general, an encoding process has the following effect on a state:

$$\alpha |0\rangle + \beta |1\rangle \mapsto \alpha |0\rangle_L + \beta |1\rangle_L$$

where $|0\rangle_L$ and $|1\rangle_L$ are called the logical(or encoded) 0 and 1 states

# The 3-qubit code

Quantum bit-flip channel: applies the X gate with probability $p$.
We can encode qubits in the following way:

$$|0\rangle \mapsto |000\rangle$$

$$|1\rangle \mapsto |111\rangle$$

On a general superposition:

$$\alpha |0\rangle + \beta |1\rangle \mapsto \alpha |000\rangle + \beta |111\rangle$$

Doesn't violate No-Cloning Theorem!

$$\alpha |000\rangle + \beta |111\rangle \neq (\alpha |0\rangle + \beta |1\rangle)^{\otimes 3}$$

# Recovery

Our state $|\psi_{enc}\rangle = \alpha|000\rangle + \beta|111\rangle$ is subject to error $U_{err}$. $U_{err}$ randomly flips at most one qubit, e.g.

$$\alpha|000\rangle + \beta|111\rangle \mapsto \alpha|010\rangle + \beta|101\rangle$$

How can we correct these errors?

# Parity Measurements

Parity measurements can give us information on the error without actually measuring the state. For the 3-qubit code, we can measure which qubits agree/disagree with each other.

$$\alpha |010\rangle + \beta |101\rangle$$

We compare the values of the first and second qubit, and the first and third qubit:

$$0 \oplus 1 = 1, 0 \oplus 0 = 0$$

# More Parity Measurements

| Final State | Parity Measurements | Recovery Operation |
|---|---|---|
| $\alpha \|000\rangle + \beta \|111\rangle$ | 00 | None |
| $\alpha \|100\rangle + \beta \|011\rangle$ | 11 | X on Qubit 1 |
| $\alpha \|010\rangle + \beta \|101\rangle$ | 10 | X on Qubit 2 |
| $\alpha \|001\rangle + \beta \|110\rangle$ | 01 | X on Qubit 3 |

# Stabilizer States

An operator A stabilizes $|\psi\rangle$ if:

$$A|\psi\rangle = |\psi\rangle$$

$|\psi\rangle$ is an eigenstate of A with eigenvalue $+1$.

# More Stabilizer States

## Definition (Stabilizer State)

An N-qubit state $|\psi\rangle_N$ is known as a stabilizer state if there exists a subgroup S of the N-manifold Pauli group $\mathcal{P}^N$ such that

$$A|\psi\rangle = |\psi\rangle \ \forall\, A \in S$$

# Generators of Stabilizer States

Stabilizer states can also be defined by their generators. For example, the state

$$|\psi\rangle = \frac{|000\rangle + |111\rangle}{\sqrt{2}}$$

is a stabilizer state defined by the generators

$$K_1 = X \otimes X \otimes X,$$

$$K_2 = Z \otimes Z \otimes I,$$

$$K_3 = I \otimes Z \otimes Z$$

# Encoding with Stabilizer States

The example 7-qubit code has the following encoded $|0\rangle$ and $|1\rangle$ states:

$$\begin{aligned} |0\rangle_L &= |0000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle \\ &+ |0001111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle \\ |1\rangle_L &= |1111111\rangle \end{aligned} \tag{1}$$

With stabilizers:

$$\begin{aligned} K_1 &= IIIXXXX, K_2 = XIXIXIX, \\ K_3 &= IXXIIXX, \quad K_4 = IIIZZZZ \\ K_5 &= ZIZIZIZ, \quad K_6 = IZZIIZZ. \end{aligned} \tag{2}$$

To encode an arbitrary state with the 7-qubit code we project the state into a $+1$ eigenstate of the stabilizer group.

# Error Correction with Stabilizer Codes

If we wish to encode an arbitrary state with the 7-qubit code we have:

$$\alpha |0\rangle + \alpha |1\rangle \mapsto \alpha |0\rangle_L + \alpha |1\rangle_L$$

which is also a $+1$ eigenstate of our generator group.
Now, if we have an error $E \in \mathcal{P}$ acting on our encoded state, we will be able to tell what type of error it is and where it occurs.
But how??

# Detecting and Correcting Errors

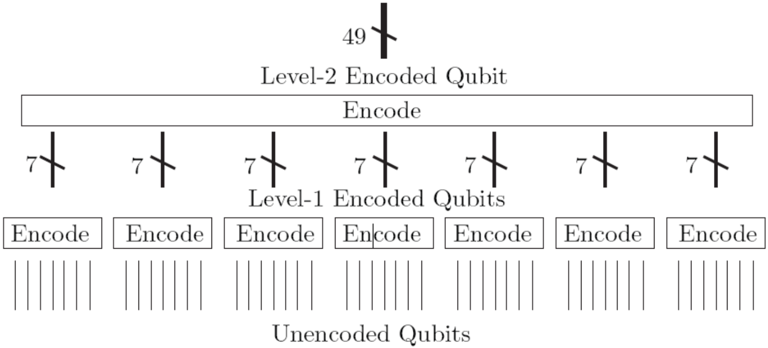
If E commutes with one of our stabilizers, it will leave the state unchanged. If it anti-commutes with a stabilizer, it will turn the state into a -1 eigenstate of the stabilizer.

E will commute and anti-commute with a unique set of generators. Once we find which generators E commutes with, we can determine what type of error E is and where it occured on our qubits.

For example, if our error is an X gate acting on the first qubit, then the error will commute with $K_5$, and anti-commute with $K_4$ and $K_6$. Similarly, an X error on the second qubit will commute with $K_6$ and anti-commute with $K_4$ and $K_5$

# Code Concatenation

If we can encode a qubit one time... why not twice?




If the probability of error on the lowest level is $p$, then after $k$ encoding levels, the probability of error occurring is $\frac{(cp)^{2^k}}{c}$

# Threshold Theorem

### Theorem (Quantum Threshold Theorem)

*A quantum circuit of $S$ gates can be built with a probability of error below $\epsilon$ with*

$$O\left(S\left(\log^m\left(\frac{S}{\epsilon}\right)\right)\right)$$

*gates on hardware that each introduce error with probability $p$ less than a constant threshold, and given reasonable information about the noise in the hardware.*