Integral Geometry

Shivaani Venkatachalam

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Introduction

Definition

Integral Geometry: Random variables, geometric quantities, invariant measures, and their connection.

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- Crofton: expected value could be used to calculate geometric quantities
- Bertrand: Bertrand's Paradox, critiqued Crofton's definition of randomness
- Poincaré: randomness could be invariant measures, "kinematic density"
- Blaschke: published many papers, progressed Integral Geometry far along

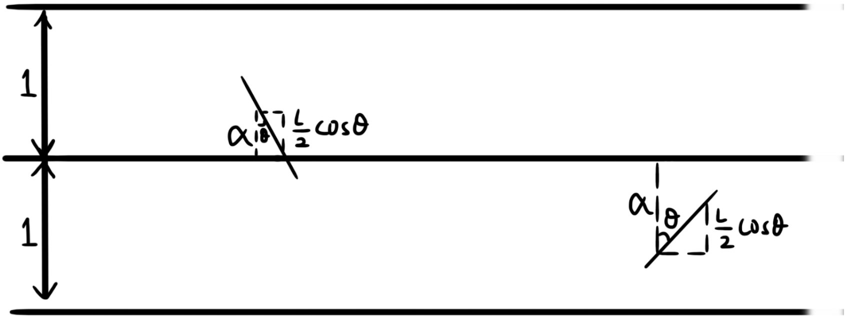
Buffon's Needle Problem

Theorem (Buffon's Needle Problem)

If we had a wooden floor made up of parallel planks of 1 unit of width, and we dropped a needle of shorter length than this width, the probability that the needle lands on a crack would be $\frac{2l}{\pi}$.

Buffon's Needle Problem (Solution)

l = length of needle $(0 < l < 1)$
α = vertical distance between center of needle and crack
θ = angle between needle and vertical distance



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As the definition of random needs to be precise, we can say that both α and θ follow a uniform distribution:

$$\alpha \sim Unif[0, \frac{1}{2}] \text{ and } \theta \sim Unif[0, \frac{\pi}{2}]$$

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The joint probability density function of (α, θ) is:

$$\begin{cases} \frac{4}{\pi} & 0 \leq \alpha \leq \frac{1}{2} \text{ and } 0 \leq \theta \leq \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$

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$$\alpha \leq \frac{l}{2}(\cos\theta).$$

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Thus, the probability the above is true is:

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{lcos\theta}{2}} \frac{4}{\pi} \; da \; d\theta = \frac{2l}{\pi}$$

and we have finished the solution.

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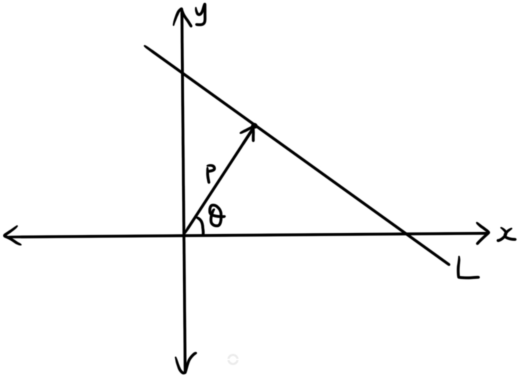
$$E[X] = E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} \frac{2l_i}{\pi} = \frac{2l}{\pi}.$$

We can define a line L by its distance from the origin, p, and the angle it forms, θ, where $0 \leq p$ and $0 \leq \theta < 2\pi$.
The equation of this line L, expressed as $L(p, \theta)$, can be written as:

$$p = \cos(\theta)x + \sin(\theta)y.$$



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M: rigid motion of a set of points; rotation of angle α and a translation by the vector (x_0, y_0):

$$x' = M(x) = x_0 + (\cos(\alpha)x - \sin(\alpha)y)$$

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From this, we can write the inverse motion as:

$$x = M^{-1}(x') = \cos(\alpha)(x' - x_0) + \sin(\alpha)(y' - y_0)$$

$$y = M^{-1}(y') = -\sin(\alpha)(x' - x_0) + \cos(\alpha)(y' - y_0)$$

Invariant Measure

Lemma

Kinematic measure is invariant under rigid motions of a set of lines. For a line defined by the coordinates (p, θ), its kinematic measure is given by:

$$dK = dp \wedge d\theta.$$

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We can express the coordinates (p', θ') as:

$$p' = p + \cos(\theta + \alpha)x_0 + \sin(\theta + \alpha)y_0$$

$$\theta' = \theta + \alpha$$

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Therefore, kinematic measure is invariant under rigid motions.

Poincare Formula of Lines

Theorem

Poincaré's Formula for Lines (1896). Let C be a piecewise C^1 curve in the plane. Then the (kinematic) measure of lines meeting C is given by

$$2L(C) = \int_{\{L : L \cap C \neq \emptyset\}} n(C \cap L)\, dK(L).$$

Convex Sets

Definition

Convex Sets: A set $\Omega \subset R^2$ is convex if for each pair of points $A, B \in \Omega$, the line segment $AB \subset \Omega$.

Sylvester's Problem

Theorem (Sylvester's Problem)

Let $\omega \subset \Omega$ be two bounded convex sets in the plane. Then the probability that a random line meets ω given that it meets Ω is

$$P = \frac{L(\partial\omega)}{L(\partial\Omega)}.$$

Bertrand's Paradox

Theorem (Bertrand Paradox)

What is the average length of a random chord of a unit circle?

Bertrand's Paradox (Solution)

If we assume uniform angle and uniform distance from the origin,

$$E(\sigma_1) = \frac{\int_{\{L:L\cap\partial\Omega\neq\emptyset\}} \sigma_1 dK}{\int_{\{L:L\cap\partial\Omega\neq\emptyset\}} dK} = \frac{\pi A(\Omega)}{L(\partial\Omega)},$$

which equals $\frac{\pi R}{2}$ when Ω is a circle.

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If we assume uniform angle and uniform point on boundary,

$$E(\sigma_1) = \frac{1}{\pi L(\partial\Omega)} \int_0^{L(\partial\Omega} \int_0^{\pi} \sigma_1 \; d\theta \; ds,$$

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If we assume two uniform random points on the boundary,

$$E(\sigma_1) = \frac{1}{(L(\partial\Omega))^2} \int_0^{L(\partial\Omega} \int_0^{L(\partial\Omega} \sigma_1 \; ds_1 \; ds_2,$$

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Further Reading

- Integral Geometry & Geometric Probability by Andrejs Treibergs
- Integral Geometry and Geometric Probability by Luis Santaló

Thank you

Thanks to Simon and my TA, Rajiv, for holding this opportunity and helping me with my paper.