INTEGRAL GEOMETRY

SHIVAANI VENKATACHALAM

Abstract. In this paper we shall go through important theorems in Integral Geometry of the plane. These include Poincaré's Formula, Sylvester's Problem, Cauchy's Formula, and Crofton's Formula, as well as solutions to Buffon's Needle Problem and the Bertrand Paradox.

1. Introduction

Integral geometry first arose from refining problem statements in geometric probabilities. One example of a problem in geometric probability is the popular Buffon's needle problem, which we will solve later on in Section 2. Created by Comte de Buffon, it was the first proposition that probability could be used to study geometry.

In the late 1800s, Morgan Crofton discovered that expected value in probability could be used to calculate geometric quantities. However, Bertrand's Paradox critiqued Crofton's definition of randomness, and it was almost dismissed entirely. In 1896, Henri Poincaré proposed that the randomness could be invariant measures, which he called kinematic density, saving it from this dismissal. In the late 1930s, Blaschke published many papers on this subject as well. As a result, much progress has been made, such as using homogenous space theory to find the measure. Integral geometry is now used to solve other geometric problems as well, such as in stochastic geometry, statistical physics, and knot curvatures. [Tng15]

In Section 2 of this paper, we will solve the Buffon's needle problem and study its implications. In Section 3, we will go over Poincaré's Formula and prove that kinematic measure is invariant under rigid motions on a set of lines. In Section 4, we will explore Sylvester's Problem, Cauchy's Formula, and expressing area in terms of a support function. Finally, in Section 5, we will explore Crofton's Formula and its corollary as well as Bertrand's Paradox.

2. Buffon's Needle Problem

Theorem 2.1. *Buffon's Needle Problem. If we had a wooden floor made up of parallel planks of 1 unit of width, and we dropped a needle of shorter length than this width, what is the probability that the needle lands on a crack?*

Proof. Let us denote the length of the needle as l, where $0 < l < 1$. The problem uses a needle to signify that we are working with a line segment with endpoints.

We can use the vertical distance between the center of the needle and the closest crack, α, and the angle between the needle and "y-axis", θ, to observe the needle's position once dropped.

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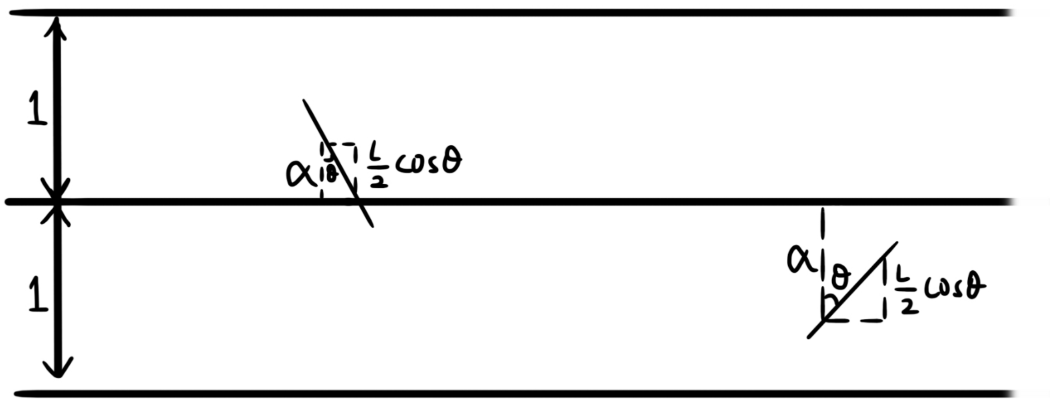


As the definition of random needs to be precise, we can say that both α and θ follow a uniform distribution:

$$\alpha \sim Unif[0, \frac{1}{2}] \; and \; \theta \sim Unif[0, \frac{\pi}{2}].$$

The joint probability density function of (α, θ) is:

$$\begin{cases} \frac{4}{\pi} & 0 \leq \alpha \leq \frac{1}{2} \; and \; 0 \leq \theta \leq \frac{\pi}{2} \\ 0 & otherwise \end{cases}$$

By a geometric analysis, the needle crosses the crack if and only if

$$\alpha \leq \frac{l}{2}(\cos\theta).$$

Thus, the probability the above is true is:

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{lcos\theta}{2}} \frac{4}{\pi} \; da \; d\theta = \frac{2l}{\pi}$$

and we have finished the solution. ■

We can observe something interesting from this result. If X denotes the number of intersections between needle and cracks, X can only be 0 or 1. Thus, the expected value of X is the same as the probability the needle intersects a crack. Thus,

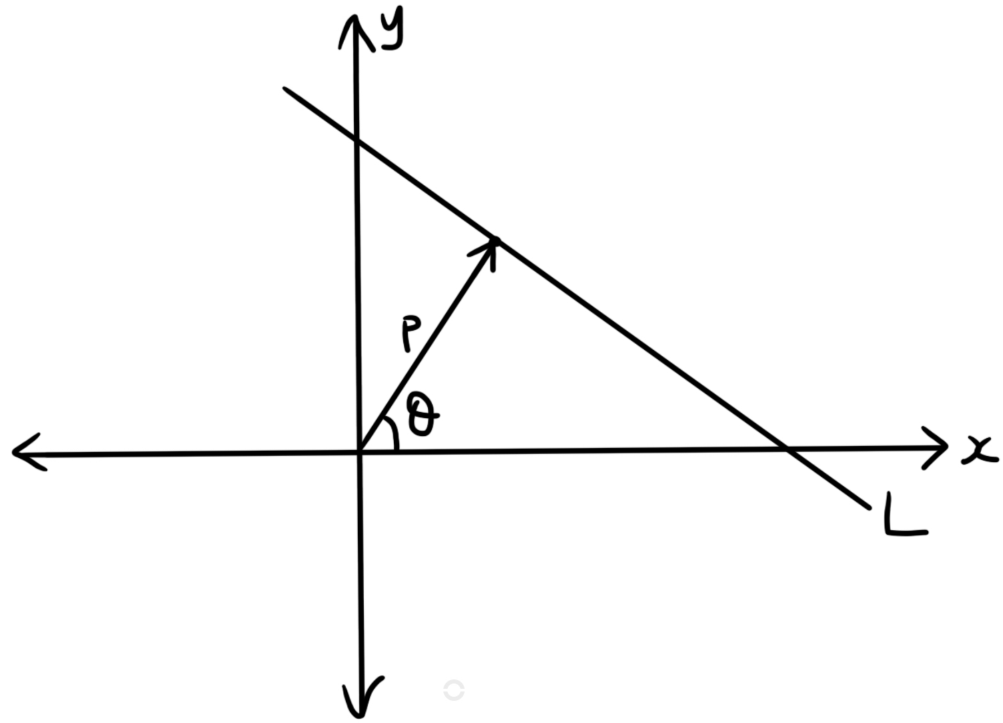
$$E[X] = \frac{2l}{\pi}.$$

We can prove that this holds true for all lengths of $l > 0$. Let us say that the needle l is made up of n shorter needles of length $l_1, l_2, \cdots, l_n$, where $0 < l_i < 1$, and $l_1+l_2+\cdots+l_n = 1$. We define X_i to be the number of intersections between needle piece l_i and the cracks, so $X_1 + X_2 + \cdots + X_n = X$. Thus,

$$E[X] = E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} \frac{2l_i}{\pi} = \frac{2l}{\pi}.$$

This result is significant because it shows that we can express the length of a needle by the average number of its intersections with the cracks. It shows we can use the expectation of a random variable in order to observe a geometric quantity.

3. Poincaré's Formula and Invariant Measures



We can define a line L by its distance from the origin, p, and the angle it forms, θ, where $0 \leq p$ and $0 \leq \theta < 2\pi$.

The equation of this line L, expressed as $L(p, \theta)$, can be written as:

$$p = \cos(\theta)x + \sin(\theta)y.$$

Rigid motions are transformations, translations, and rotations of a set where the distance between points does not change.

We define the rigid motion M of a set of points as a rotation of angle α and a translation by the vector (x_0, y_0):

$$x' = M(x) = x_0 + (\cos(\alpha)x - \sin(\alpha)y)$$
$$y' = M(y) = y_0 + (\sin(\alpha)x + \cos(\alpha)y)$$

From this, we can write the inverse motion as:

$$x = M^{-1}(x') = \cos(\alpha)(x' - x_0) + \sin(\alpha)(y' - y_0)$$
$$y = M^{-1}(y') = -\sin(\alpha)(x' - x_0) + \cos(\alpha)(y' - y_0)$$

Lemma 3.1. *Kinematic measure is invariant under rigid motions of a set of lines. For a line defined by the coordinates (p, θ), its kinematic measure is given by:*

$$dK = dp \wedge d\theta.$$

We can define a measure as a generalization of the notion of a geometric quantity (length, area, volume), mass, or probability.

Proof. We begin by observing how a rigid motion M affects a line $L(p, \theta)$. Using our equations from earlier:

$$p = \cos(\theta)x + \sin(\theta)y$$

$= \cos(\theta)[\cos(\alpha)(x' - x_0) + \sin(\alpha)(y' - y_0)] + \sin(\theta)[-\sin(\alpha)(x' - x_0) + \cos(\alpha)(y' - y_0)]$

Factoring out,

$$= [\cos(\theta)\cos(\alpha) - \sin(\theta)\sin(\alpha)](x' - x_0) + [\cos(\theta)\sin(\alpha) + \sin(\theta)\cos(\alpha)](y' - y_0).$$

Using trigonometry sum identities,

$$= \cos(\theta + \alpha)(x' - x_0) + \sin(\theta + \alpha)(y' - y_0).$$

Therefore, we can write the equation of a line L' as:

$$p + \cos(\theta + \alpha)x_0 + \sin(\theta + \alpha)y_0 = \cos(\theta + \alpha)x' + \sin(\theta + \alpha)y'.$$

We can express the coordinates (p', θ') as:

$$p' = p + \cos(\theta + \alpha)x_0 + \sin(\theta + \alpha)y_0$$

$$\theta' = \theta + \alpha$$

. Then, the Jacobian formula for change in measure is

$$dp' \wedge d\theta' = |J| dp \wedge d\theta$$

.

Therefore, kinematic measure is invariant under rigid motions. ■

Theorem 3.2. *Poincaré's Formula for Lines (1896). Let C be a piecewise C^1 curve in the plane. Then the (kinematic) measure of unoriented lines meeting C, counted with multiplicity, is given by*

$$2L(C) = \int_{\{L: L \cap C \neq \emptyset\}} n(C \cap L)\ dK(L).$$

[Tre]

Here, $n(C \cap L)$ represents the number of times the curve intersects the lines. $L(C)$ denotes the length of the curve and $dk(L)$ refers to its kinematic measure.

We will use this formula more in upcoming sections.

4. Sylvester's Problem, Cauchy's Formula, and Area by Support Function

We can expand Poincaré's Formula to hold for convex sets.

Definition 4.1. Convex Set: A set $\Omega \subset R^2$ is convex if for each pair of points $A, B \in \Omega$, the line segment $AB \subset \Omega$.

As $n(L \cap \Omega)$ is either 0 or 2, the measure of unoriented lines that meet a convex set is given by

$$L(\partial\Omega) = \int_{\{L: L \cap \Omega \neq \emptyset\}} dK.$$

Furthermore, we can define the conditional probability of an event A occurring given event B as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

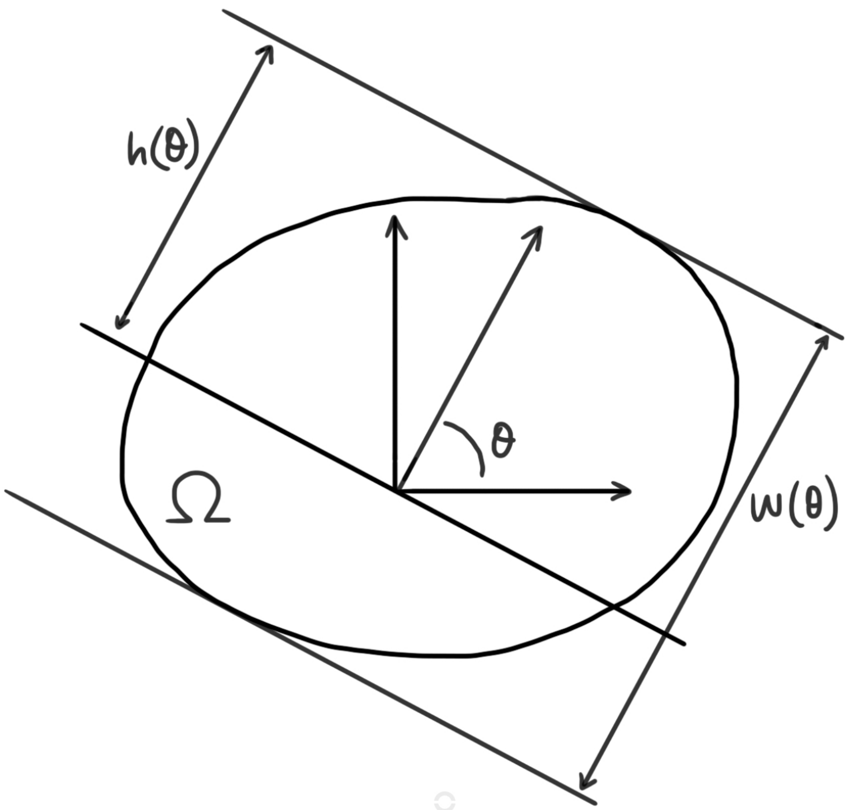
Theorem 4.2. *Sylvester's Problem. Let $\omega \subset \Omega$ be two bounded convex sets in the plane. Then the probability that a random line meets ω given that it meets Ω is*

$$P = \frac{L(\partial\omega)}{L(\partial\Omega}.$$

[Tre]

Corollary 4.3. *Let C be a piecewise C^1 curve contained in a compact convex set Ω. Of all random lines that meet Ω, the expected number of intersections with with C is*

$$E(n) = \frac{2L(C)}{L(\partial\Omega)}.$$



For $0 \leq \theta < 2\pi$ in the figure above, the support function, $h(\theta)$, can be defined as the largest p such that the line L with coordinates (p, θ) does intersect with bounded convex domain Ω. The width $w(\theta) = h(\theta) + h(\theta + \pi)$.

Corollary 4.4. *Cauchy's Formula. Let Ω be a bounded convex domain. Then*

$$L(\partial\Omega) = \int_0^{2\pi} h(\theta)\ d\theta = \int_0^{\pi} w(\theta)\ d\theta.$$

[Tre]

Proof. By the formula for convex sets,

$$L(\partial\Omega) = \int_{\{L: L\cap\Omega\neq\emptyset\}} dK.$$

Since $dK = dp \wedge d\theta$, $0 \leq \theta < 2\pi$, and p can range from 0 to $h(\theta)$ by definition,

$$L(\partial\Omega) = \int_{\{L:L\cap\Omega\neq\emptyset\}} dK = \int_0^{2\pi}\int_0^{h(\theta)} dp\ d\theta$$

Simplifying this,

$$\begin{aligned} L(\partial\Omega) &= \int_{\{L:L\cap\Omega\neq\emptyset\}} dK = \int_0^{2\pi}\int_0^{h(\theta)} dp\ d\theta \\ &= \int_0^{2\pi} h(\theta)d\theta = \int_0^{\pi} h(\theta) + h(\theta+\pi)\ d\theta \\ &= \int_0^{\pi} w(\theta)d\theta. \end{aligned}$$

from our definition of $w(\theta)$. ■

Additionally, we can express the area of the boundary in terms of a support function

Theorem 4.5. *Suppose Ω is a compact, convex domain with a C^2 boundary. Then*

$$A(\Omega) = \frac{1}{2}\int_0^{2\pi} h ds = \frac{1}{2}\int_0^{2\pi} h(h+\ddot{h})d\theta$$

5. Crofton's Formula and the Bertrand Paradox

Crofton's Formula was significant in that it used the expectation of intersections with a random line to calculate the length of a plane curve.

Theorem 5.1. *Crofton's Formula (1868). Let $D \subset R^2$ be a domain with compact closure, $L \subset R^2$ a random line and $\sigma_1(L\cap D)$ be the length (one-dimensional measure). Then,*

$$\pi A(D) = \int_{\{L:L\cap D\neq\emptyset\}} \sigma_1(L\cap D)dK(L).$$

[Tre]

We also have its corollary.

Corollary 5.2. *Crofton (1885). Let Ω be a bounded convex domain in the plane. Then the probability that two random lines intersect in Ω given that they both meet Ω is*

$$P = \frac{2\pi A(\Omega)}{L(\partial\Omega)^2}.$$

[Tre]

Using the isoperimetric inequality, which states $L^2 \geq 4\pi A$, if L is the length of a closed curve and A is its area,

$$4\pi A(\Omega) \leq L(\partial\Omega)^2$$

This inequality is equal only if the domain is a circle. Thus, $P \leq \frac{1}{2}$.

Finally, the Bertrand Paradox.

Theorem 5.3. *Bertrand Paradox. What is the average length of a random chord of a unit circle?*

There are many solutions to this problem, depending on the definition of random.

If we assume uniform angle and uniform distance from the origin,

$$E(\sigma_1) = \frac{\int_{\{L:L\cap\partial\Omega\neq\emptyset\}} \sigma_1 dK}{\int_{\{L:L\cap\partial\Omega\neq\emptyset\}} dK} = \frac{\pi A(\Omega)}{L(\partial\Omega)},$$

which equals $\frac{\pi R}{2}$ when Ω is a circle.

If we assume uniform angle and uniform point on boundary,

$$E(\sigma_1) = \frac{1}{\pi L(\partial\Omega)} \int_0^{L(\partial\Omega} \int_0^{\pi} \sigma_1 \; d\theta \; ds,$$

which equals $\frac{4R}{\pi}$ when Ω is a circle.

If we assume two uniform random points on the boundary,

$$E(\sigma_1) = \frac{1}{(L(\partial\Omega))^2} \int_0^{L(\partial\Omega} \int_0^{L(\partial\Omega} \sigma_1 \; ds_1 \; ds_2,$$

which equals $\frac{4R}{\pi}$ when Ω is a circle.

As there are different solutions depending on the definition of random, it is important to make this clear as we did in our solution of Buffon's needle.

References

[Tng15] Barry Jia Hao Tng. How big is that cookie? the integral geometric approach to geometrical quantities. 2015.

[Tre] Andrejs Treibergs. Integral geometry & geometric probability.