Posets and their Incidence Algebras

Sarthak Bal Mitra
balmitra.sarthak@gmail.com

Euler Circle

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What is a Poset?

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2. Transitivity: If $x \leq y$ and $y \leq z$, then $x \leq z$ for $x, y, z \in S$
3. Anti-Symmetry: If $x \leq y$ and $y \leq x$, then $x = y$ for $x, y \in S$

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S is any set of positive integers and $x \leq y$ if x is a factor of y. Notice that the relation satisfies all three properties.

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S is any set of positive integers and $x \leq y$ if x is a factor of y. Notice that the relation satisfies all three properties. Now, $2 \leq 6$ and $3 \leq 6$, they are called *comparable* elements. 2 is not a factor of 3 and vice versa. They are known as *incomparable* elements. Think of comparable as "orderable". This is the meaning of PARTIAL ORDER.

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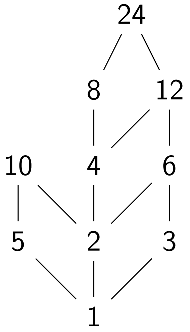
N for Nike

The following is an example of a Hasse Diagram, let S be the union of the factors of 10 and 24:
$\{1,2,5,10,3,4,6,8,12,24\}$ and $x \leq y$ if x is a factor of y.



We will refer to this poset and diagram as N (for Nike since it looks like a swoosh after tilting clockwise)

Intervals of a Poset

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The Incidence Algebra

The INCIDENCE ALGEBRA $\mathrm{I(S)}$ of S is the set of all functions from $Int(S)$ to a field K. If g is one such function, we use $g(x, y)$ to mean $g([x, y])$.

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This is exactly the same as the multiplication of matrices, more specifically matrices which are *upper triangular*, meaning all values below the main diagonal (top left to bottom right) are 0. We will see this more clearly with an example later.

The Identity Function

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This is only non-zero when $z = y$. Hence $f\delta(x, y) = f(x, y)$. This function in fact corresponds to the identity matrix.

The Zeta Function

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Simple. We will skip some of the applications of this function but feel free to read my paper to learn more about the number of partitions in posets.

Now, does the Zeta function have an inverse? We know that $\zeta(x,x) = 1$, and the matrix representation of Zeta is upper triangular so the determinant is the product of the main diagonal which is just 1. Hence the inverse exists.

The Mobius Function

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On a poset S, it is denoted by (μ or) $\mu_S = \zeta^{-1}$. We can also see it as $\mu \cdot \zeta = \delta$. We notice this recursive definition as well. For a locally finite poset S and $[x, y] \in Int(S)$,

$\mu(x, x) = 1$ and $\mu(x, y) = -\sum_{x \leq z < y} \mu(x, z)$

The Mobius Inversion Theorem

Theorem

Let S be a poset with finite principal ideals. If f and g are functions from S to a field K, then:

$$g(y) = \sum_{x \leq y} f(x) \Rightarrow f(y) = \sum_{x \leq y} g(x)\mu(x, y)$$

The Mobius Inversion Theorem

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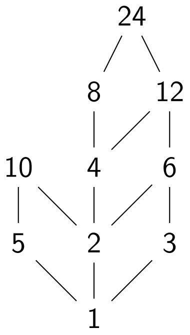
$$\sum_{z \leq y} f(z)\delta(z,y) \iff f(y)$$

Notice the following: $g(x) = \sum_{z \leq x} f(z)$ and $\sum_{z \leq x \leq y} \mu(x,y) = \delta(z,y)$

∎

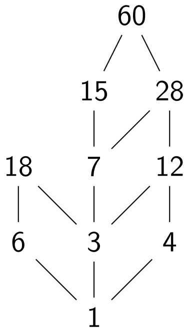
The Classic Application

Let's now take N (or Nike) from page 6, and make the following transformation from N to N'. Let $x_i \in N$ and $y_i \in N'$. Then I choose an arbitrary ordering $x_1 = 1, x_2 = 2, x_3 = 5, x_4 = 10, x_5 = 3, x_6 = 4, x_7 = 6, x_8 = 8, x_9 = 12, x_{10} = 24$. (as long as I am consistent throughout I can pick any ordering). y_i is the sum of all x_j for which $x_j \leq x_i$.



Transformed N

This is analogous to $g(y) = \sum_{x \leq y} f(x)$. So formally, if X is the row vector with x_i (or $f(x)$) and Y is the row vector with y_i (or $g(y)$), then for the Zeta Matrix Z of N, $Y = XZ$



The Zeta Matrix (not the movie)

We use X: [1,2,5,10,3,4,6,8,12,24] to represent the rows and columns and use the Zeta function to input 1's or 0's into the matrix. So for example if we want y_{10}, we sum the 10th column: $1(1) + 2(1) + 5(0) + 10(0) + 3(1) + 4(1) + 6(1) + 8(1) + 12(1) + 24(1) = 60$. Now we find the matrix M so that $YM = XZM = X$.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The Mobius Matrix

We use Y: [1,3,6,18,4,7,12,15,28,60] and the mobius function to input 1's, 0's or -1's into the matrix. The mobius function for this case is defined as: $\mu(y, x) = (-1)^k$ if y/x is a product of k distinct primes and 0 otherwise. This is the CLASSICAL mobius function. So we get our desired inverse matrix.

$$\begin{bmatrix} 1 & -1 & -1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Simple proof of Mobius Inversion Theorem

If we treat the functions g, f as vectors G, F, then $G = FZ \Rightarrow GM = F$, where Z and M are the Zeta and Mobius Matrices of the poset respectively. The proof is simply stating that $ZM = I$ (the identity matrix).

Thank You!

I thank the whole Euler Circle community, including the guest lecturers for having such interesting talks. Also want to thank Simon and Sawyer specially for guiding me through this.