POSETS AND THE APPLICATIONS OF THE MÖBIUS FUNCTION

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Abstract

This paper discusses the world of partially ordered sets (posets) and looking at math from their perspective. We talk about various partitioning theorems of posets and also discuss operations on posets. The next main topic is the incidence algebra of posets. We lay down many important functions which lead to significant results. One of these functions is the Möbius function which leads to the very important Möbius Inversion Theorem. We then come to the main part of the paper in which we talk about two important applications of the Möbius Inversion Theorem.

Introduction

We start off talking about about the meaning of partial order and how Hasse diagrams represent this seemingly complex relation easily. Hasse diagrams are named after Helmut Hasse but they existed long before his work. They were named after him because of how elegantly and extensively he used these diagrams.

We define chains, antichains and chain covers, which are essential to understand Dilworth's theorem, a major theorem frequently brought up in combinatorial analysis.

Theorem 0.1. *Dilworth's Theorem: Let S be any finite partially ordered set. The size of any maximum antichain is equal to the number of chains in a minimal chain cover of S.*

Next, we have a relatively small section on special kinds of poset called Lattices. They can be defined individually as well with the help of fundamental laws. We now, build towards the applications of the Möbius inversion formula by introducing incidence algebras. I think of them to be houses in which all the magic regarding Möbius functions happens.

Definition 0.2. The INCIDENCE ALGEBRA I(S) of S is the set of all functions from $Int(S)$ to a field K, where $Int(S)$ is the set of all closed intervals of a poset.

We look at the Identity function of this algebra, and then the Zeta and Möbius functions. Next, we have:

Theorem 0.3. *The Möbius Inversion Theorem: Let S be a poset with finite principal ideals. If f and g are functions from S to a field K, and:*

$$g(y) = \sum_{x \leq y} f(x), \text{ then } f(y) = \sum_{x \leq y} g(x)\mu(x, y)$$

Proposition 0.4. *The Möbius Inversion Theorem leads to the Fundamental Theorem of Calculus in terms of Integration being analogous to a summation function and a Derivative being analogous to Möbius Inversion.*

Proposition 0.5. *The Möbius Inversion Theorem for a poset of positive integers under divisibility can be represented by a set of upper triangular matrices, which leads to a concise proof of the Möbius Inversion Theorem.*

1. What are Posets?

Definition 1.1. A partially ordered set or poset is a pair $(S, \leq)$ where S is simply a set and $\leq$ is a relation on the elements of S. $\leq$ represents any relation that satisfies these properties:

1) Reflexive: $x \leq x$ for all $x \in S$
2) Transitive: If $x \leq y$ and $y \leq z$, then $x \leq z$ for $x, y, z \in S$
3) Anti-Symmetric: If $x \leq y$ and $y \leq x$, then $x = y$ for $x, y \in S$

You may be wondering why it's known as a *partially* ordered set. This is because the relation $\leq$ can be such that $x \nleq y$ *and* $y \nleq x$ for $x, y \in S$. These numbers are known as *incomparable* numbers. If either of the relations are satisfied then they are known as *comparable* numbers. Looking at some examples will help us understand posets better.

Example 1.1:S is any set of positive integers and $x \leq y$ if x is a factor of y. The relation satisfies all three properties. So now, $2 \leq 6$ and $3 \leq 6$, but 2 is not a factor of 3 and vice versa making them incomparable. Think of comparable as "orderable". This is the meaning of PARTIAL ORDER.

Example 1.2: Let $N = \{1, 2, \ldots, n\}$ be a set. Then B_n, known as the Boolean Algebra of degree n, is the partially ordered set with S being a set containing all subsets of N and $x \leq y$ if $x \subseteq y$, where x, y are sets in S.

Example 1.3: This last example is an unusual one. S is a set of integers and $x \leq y$ if $y - x$ is a non-negative integer (the usual meaning of $\leq$). In this, all elements are comparable to each other, making it a *totally ordered set. But a totally ordered set is also a partially ordered set.* This set can be represented by a sequence $a_1 < a_2 < \ldots < a_n$.

Definition 1.2. A poset is known as a CHAIN if all pairs of elements are comparable. You can think of chains as sequences (like the example above).

Definition 1.3. Conversely, a poset in which no pairs of elements are comparable is known as an ANTICHAIN. In Example 1.1, if we consider the set of all primes, we get an antichain.

Like sets, posets can be infinite or finite.

2. Hasse Diagrams

One interesting thing is the modeling of posets. How do we represent them? We saw that it is easy to represent a totally ordered set but what about strictly-partially ordered sets. This is done by *Hasse diagrams.* Consider the following process:

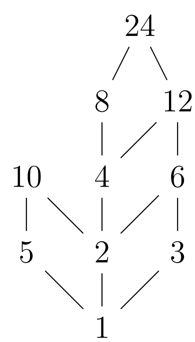
Make a graph in which all the vertices represent elements of a poset. Then make a directed edge from x to y if $x \leq y$ (for distinct x, y). Consider these as"one-way roads" and "cities".

Let's say I want to go from City P to City Q. For this, P has to be $\leq$ Q so there exists a one-way road from P to Q. But I enjoy photography of different places so I want to pass through as many cities as I can on my way. So if there exists another route, I delete the "road" or **edge** from P to Q. Now I do this for all P and Q belonging to the poset. This simplifies the diagram. Now, we arrange the vertices such that all the edges are directed upwards. This would enable us the get rid of the arrows of the directed edges and our final result is a *Hasse diagram.*

We can also define a Hasse diagram more formally. For a poset S and $x, y \in S$, we say y covers x if $x < y$ and there does not exist $z \in S$ for which $x < z$ and $z < y$. So a Hasse diagram is just a graph (with the vertices representing all the elements) where we draw an upward edge (can be diagonal but upward too) from x to y if y covers x for all $x, y \in S$.

Example 2.1: Here is an example of a Hasse diagram. We will call it N (for NIKE). I like to call it NIKE since if rotated clockwise, it looks like a swoosh.

Let S be the union of the factors of 10 and 24: {1,2,5,10,3,4,6,8,12,24} and $x \leq y$ if x is a factor of y. Then we get the following:



3. Dilworth's Theorem

Definition 3.1. A chain cover of a poset is a collection of chains, whose union is the poset itself.

Theorem 3.2. *Dilworth's Theorem: Let S be any finite partially ordered set. The size of any maximum antichain is equal to the number of chains in a minimal chain cover of S.*

Proof. First we prove that if the minimal chain cover of S has w elements, then the size of the maximum antichain is w. For this, we just take one element from each of the w chains, which will form an antichain. The antichain cannot have more than w elements since the $w + 1^{\text{st}}$ element will be comparable to some other element in the antichain.

Now we will prove that if w is the size of a maximum antichain, then you can partition the poset into w chains. Let the size of the poset be q. We induct on q. The base case ($q = 1$) is easy to see. Now, for some $q > 1$, assume it is true for all positive integers till $q - 1$. We now prove it for q. We partition the set into 3 subsets. The first is the maximum antichain itself, which has w elements, denoted by W. Let w_i be any element belonging to the maximum antichain. The second subset U, has all the elements b such that $b > w_i$. Finally, the third subset D has all the elements j such that $j < w_i$. Note that the union of these three subsets makes up S.

CASE 1: D and U are both non-empty.

Consider, $S - D = W \cup U$. This is a poset with $< q$ elements (since D is non-empty. This poset also has a maximum antichain with w elements so according to the inductive hypothesis, we can partition it into w chains. Note that every w_i will be"connected" to a different chain. Now we consider $S - U = W \cup D$ and do the exact same thing and partition it into w chains. So now, for every w_i we have two chains: one with w_i being the minimal element and the other with w_i being the maximal element. We now connect or link these two chains so that out entire poset S is now partitioned into w chains.

CASE 2: 1 or more of D and U are empty.

Now let f_1 and f_2 be the maximal element and minimal element of S respectively. Note that one of them has to be part of the antichain according to our case condition. (In the case that both D and U are empty, $f_1 = f_2 = w_i$). Let C_f be the max length chain which has both f_1 and f_2. Now $S - C_f$ has $< q$ elements and has a maximal antichain with $w - 1$ elements so by our inductive hypothesis, it can be partitioned into $w - 1$ chains. Our wth chain is C_f. And we have successfully partitioned our poset into w chains. ■

4. Lattices

Before we define what a lattice is, we need some important terminology.

Definition 4.1. For a poset S, and $x, y \in S$, let j be the lowest element such that $x \leq j$ and $y \leq j$. This is known as the JOIN of x and y and is the common minimum upper bound of x and y. It is denoted by $x \vee y$.

Definition 4.2. Similarly let m be the highest element such that $m \leq x$ and $m \leq y$. This is known as the MEET of x and y and is the common maximum lower bound of x and y. It is denoted by $x \wedge y$.

Definition 4.3. A LATTICE is a poset in which all pairs of elements have a join and a meet.

Following from the definition, we have that every finite lattice has to have a single maximum element and a single minimum element. Let **1** denote the max element and **0** denote the minimum element.

A lattice can also be defined in a different way. Not in terms of a poset. A set S is a lattice if with two binary operations $\vee$ and $\wedge$, the following axioms are satisfied.

Associativity: $x \vee (y \vee z) = (x \vee y) \vee z$ and $x \wedge (y \wedge z) = (x \wedge y) \wedge z$

Commutativity: $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$

Absorbtion Law: $x \vee (x \wedge y) = x = x \wedge (x \vee y)$

A set with those binary operations and two elements $\mathbf{0}, \mathbf{1} \in S$ is called a *distributive* lattice if the following axioms are further satisfied.

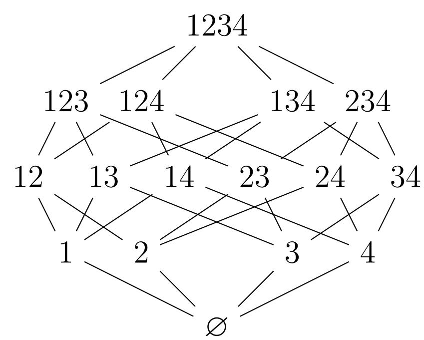
$x \vee \mathbf{1} = \mathbf{1}$ and $x \wedge \mathbf{1} = x$

$x \vee \mathbf{0} = x$ and $x \wedge \mathbf{0} = \mathbf{0}$

Idempotent Law: $x \vee x = x \wedge x = x$, this follows from the Absorbtion Law.

Now that we have some knowledge on lattices, let's see some examples.

Example 2.1: The boolean algebra of degree n is a lattice. We can see this by observing that the join of two elements would be their union and the meet would be their intersection. Here is the Hasse diagram of the boolean algebra of degree 4.



5. Operations on Posets

We begin this section by defining isomorphism:

Definition 5.1. S and P are ISOMORPHIC if there is a one-one onto map $f : S \to P$ and $x \leq y$ if and only if $f(x) \leq f(y)$. For two posets S and P, I will use $S \cong P$ to show isomorphism.

I think it is a nice observation that two posets are isomorphic if and only if their Hasse diagrams are the same. Here are some operations we can perform on posets:

Definition 5.2. If S and P are two posets, $S + P$ is the *direct sum* or *disjoint union* of these posets. The set is $S \cup P$ and $\leq$ is defined as such: For $x, y \in S + P$, we say $x \leq y$ if $x, y \in S$ (or P) and $x \leq y$ in S (or P).

A poset which cannot be broken down into the *disjoint union* of two non-empty posets is called CONNECTED. Otherwise it is called DISCONNECTED.

The disjoint union of S with itself n times is denoted by nS. Notice that an antichain of length $n \cong n\mathbf{1}$ (a one element poset).

Definition 5.3. The poset $S \oplus P$ is called the *ordinal sum* of S and P. The set is again $S \cup P$ and $\leq$ is defined as such. For $x, y \in S \oplus P$, we say $x \leq y$ if $x, y \in S$ (or P) and $x \leq y$ in S (or P) or, $x \in S$ and $y \in P$.

Notice that a chain of length $n \cong \mathbf{1} \oplus \mathbf{1} \oplus \ldots \oplus \mathbf{1}$ (n times). SERIES PARALLEL POSETS are posets which can be constructed by a combination of direct sums and ordinal sums of the one element poset.

Definition 5.4. If S and P are posets, then SP is the direct (or Cartesian) product of P and Q. It is defined on the set $\{(x, y) : x \in S \text{ and } y \in Q\}$ such that $(x, y) \leq (x_2, y_2)$ if $x \leq x_2$ in S and $y \leq y_2$ in P.

The Cartesian product of S, n times with itself is denoted by S^n.

6. Incidence Algebras of Posets

For a poset S, let x and $y \in S$. The set of all z such that $x \leq z \leq y$ is known as the CLOSED INTERVAL between x and y, it is denoted by $[x, y]$. S is called locally finite if all intervals of S are finite. S can be locally finite and infinite at the same time. $Int(S)$ is the set of all closed intervals of S.

Definition 6.1. The INCIDENCE ALGEBRA I(S) of S is the set of all functions from $Int(S)$ to a field K. If g is one such function, we use $g(x, y)$ to mean $g([x, y])$. Multiplication of functions is defined by

$$(f \cdot g)(x, y) = \sum_{x \leq z \leq z} f(x, z) g(z, y)$$

This is exactly the same as the multiplication of matrices, more specifically matrices which are *upper triangular*, meaning all values below the main diagonal (top left to bottom right) are 0. We will see this more clearly with an example later.

The first question is do we have an identity function? That is a function u such that $fu = uf = f$ for all $f \in I(S)$. Consider the function:

$\delta(x, y) = 1$ if $x = y$ and 0 otherwise. From the definition of multiplication of functions given above we can see how this is the identity function. In fact it corresponds to the identity matrix. The next function is also a simple zero-one function but it is very useful.

Definition 6.2. The ZETA function $\in I(S)$ is defined to be $\zeta(x, y) = 1$ if $x \leq y$ and 0 otherwise.

Definition 6.3. A MULTICHAIN in a poset is a multiset of elements $x_1, x_2, \ldots, x_m$ satisfying $x_1 \leq x_2 \leq \ldots \leq x_m$. The difference between multichains and chains is that in multichains, elements don't have to be distinct.

Proposition 6.4. *From [Bóna, 2006] Let S be a locally finite poset and $x \leq y$ be two elements. Then the number of multichains $x = x_0 \leq x_1 \leq \ldots \leq x_k = y$ is equal to $\zeta^k(x, y)$*

Proof. We induct on k. For the base case we have $k = 1 \Rightarrow \zeta^1(x, y) = 1$ if and only if $x \leq y$. (This also works for $k = 0$ but the $\leq$ changes to $=$). Now we assume the proposition is true for all positive integers till $k - 1$ and now prove it for k.

A multichain $x = x_0 \leq x_1 \leq \ldots \leq x_k = y$ can be decomposed to two multichains:
1) $x = x_0 \leq x_1 \leq \ldots \leq x_{k-1} = z$
2) $z \leq y$
Our induction hypothesis tells us that the number of multichains of type 1) is $\zeta^{k-1}(x, z)$ and of type 2) is simply $\zeta(z, y)$

If we sum over all possible $z \in [x, y]$, we get that the total number of multichains $x = x_0 \leq x_1 \leq \ldots \leq x_k = y$ is equal to:

$$\sum_{z\in[x,y]} \zeta^{k-1}(x,z)\cdot\zeta(z,y) = \zeta^k(x,y)$$

■

This basically tells us that a better definition of the length of a chain or multichain is the number of times $\leq$ occurs, which is equal to the number of elements minus one.

A justification is that when we walk up the Hasse diagram of a poset, we make "the length" number of steps.

Proposition 6.5. *Let $[x,y]\in Int(S)$. Then the number of chains of length k that start at x and end in y is $(\zeta-\delta)^k(x,y)$.*

The proof of this is analogous to the proof of the previous Proposition.

The next question we ask is: is there an inverse of the Zeta function? Well, the Zeta function can also be represented by matrix, which is also upper triangular and $\zeta(x,x) = 1$ so the determinant of this matrix = product of main diagonal = 1. Hence, its inverse exists. This brings us to a very important function: the MÖBIUS function.

Definition 6.6. The MÖBIUS function is the inverse of the Zeta function. On a poset S, it is denoted by (μ or) $\mu_S = \zeta^{-1}$. We can also see it as $\mu\cdot\zeta = \delta$. We notice this recursive definition as well. For a locally finite poset S and $[x,y]\in Int(S)$,

$$\mu(x,x) = 1 \text{ and } \mu(x,y) = -\sum_{x\leq z<y}\mu(x,z)$$

This brings us to the *Möbius Inversion Theorem*

Theorem 6.7. *The Möbius Inversion Theorem: Let S be a poset with finite principal ideals. If f and g are functions from S to a field K, and:*

$$g(y) = \sum_{x\leq y} f(x), \textit{ then } f(y) = \sum_{x\leq y} g(x)\mu(x,y)$$

Proof. We see the following proof from [Stanley, 2011]

$$\begin{gathered}\sum_{x\leq y} g(x)\mu(x,y) \iff \\ \sum_{x\leq y}\mu(x,y)(\sum_{z\leq x} f(z)) \iff \\ \sum_{z\leq y} f(z)\sum_{z\leq x\leq y}\mu(x,y) \iff \\ \sum_{z\leq y} f(z)\delta(z,y) \iff \\ f(y)\end{gathered}$$

■

Notice the following critical steps: $g(x) = \sum_{z\leq x} f(z)$ (from our if statement) and

$$\sum_{z\leq x\leq y} \mu(x,y) = \delta(z,y) \text{ (from the recursive definition of the Möbius function)}$$

7. Applications of the Möbius Inversion Theorem

We will see that for posets in which the Möbius function is easy to compute, the Möbius inversion theorem helps us gain a lot of insight into the partial order relation. For example, in IEP, the inclusion - exclusion theorem, the coefficient of the subsets to add or substract are decided by the Möbius function.

Let's take a look at a **fundamental** example.

Let S be the chain of Natural Numbers under the regular meaning of $\leq$. Then,

$$\mu(x,x) = 1.$$

Now from the recursive definition of the Möbius function, we see that:

$$\mu(x,x+1) = -\mu(x,x) = -1 \text{ and } \mu(x,x+k) = 0 \text{ for } k > 1$$

Recall from calculus $\int_a^b f(x)dx$ is the area under the curve of $f(x)$, and this area can be represented by taking the summation of the area of the rectangles formed along the function. Now, imagine for some $f(x)$, you take the sum of rectangles with base length 1. This would correspond to integers or in our case, the natural numbers. So then,

$$\int_0^b f(x)dx = \sum_{k=0}^{b} f(k)$$

If we let $g(x) = \int_0^b f(x)dx$ then we get our initial conditions for the Möbius inversion theorem!

$$f(x) = \sum_{k=0}^{b} g(k)\mu(k,b) =$$

$$\sum_{k=0}^{b-2} g(k)\mu(k,b) + g(b-1)\mu(b-1,b) + g(b)\mu(b,b) =$$

$$0 + \cdots + 0 - g(b-1) + g(b)$$

$$\Rightarrow$$

$$f(x) = \frac{g(b)-g(b-1)}{b-(b-1)}$$

This is like a derivative, in the sense that we are undoing integration and we have sort of derived the **fundamental theorem of calculus** from the world of posets. Instead of Natural Numbers, we could always take some infinitesimally small value Δ and create a chain $\cdots < x-\Delta < x < x+\Delta < \dots$ to be more accurate.

It's not always this easy to find the Möbius function for Posets, but the following result allows us to get the Möbius functions for complex posets from simpler posets.

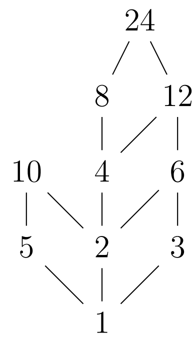
Theorem 7.1. *The Product Theorem. Let P and Q be locally finite posets, and let $P \times Q$ be their direct (or cartesian) product. If $(s,t) \leq (s',t')$ in $P \times Q$, then:*

$$\mu_{P\times Q}((s,t),(s',t')) = \mu_P(s,s')\mu_Q(t,t')$$

Proof. $$\sum_{(s,t)\leq(u,v)\leq(s',t')} \mu_P(s,u)\mu_Q(t,v) = (\sum_{s\leq u\leq s'} \mu_P(s,u))(\sum_{t\leq v\leq t'} \mu_Q(t,v)) = \delta_{(s,s')}\delta_{(t,t')} = \delta_{(s,t),(s',t')}$$

Therefore, the function $\mu_P(s,s')\mu_Q(t,t')$ is the unique function defined on $Int(P\times Q)$ that sums to 0 for non trivial intervals and 1 for trivial intervals of $P\times Q$. It is the Möbius Function. ■

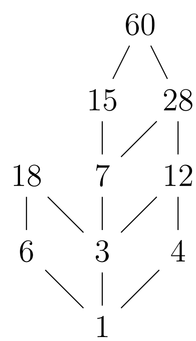
Let's now look at the poset N from *Example 2.1.* We make the following transformation from N to N'.



Let $x_i \in N$ and $y_i \in N'$. Then I choose an arbitrary ordering $x_1 = 1, x_2 = 2, x_3 = 5, x_4 = 10, x_5 = 3, x_6 = 4, x_7 = 6, x_8 = 8, x_9 = 12, x_{10} = 24$. As long as I am consistent with the ordering whenever representing this poset, I can pick any ordering.

Each y_i is the sum of all x_j for which $x_j \leq x_i$. This is analogous to $g(y) = \sum_{x\leq y} f(x)$.

We get the following:



So formally, X is the row vector with x_i or $f(x)$ and we use the same ordering to represent the rows and columns of the transformation matrix, which turns out to be the Zeta Matrix. We use the Zeta function to input 1's or 0's into the matrix. $\zeta(x,y)=1$ if x is a divisor of y and 0 otherwise.

$$\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\
0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1\\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

Let this matrix be represented by Z, then, $XZ = Y$ where,

X: [1,2,5,10,3,4,6,8,12,24] and Y: [1,3,6,18,4,7,12,15,28,60].

So for example if we want y_{10}, we multiply the 10th column with X: 1(1) + 2(1) + 5(0) + 10(0) + 3(1) + 4(1) + 6(1) + 8(1) + 12(1) + 24(1) = 60.

Now we find the matrix M so that $YM = X(ZM) = X$. For this, we use the Möbius function to input 1's, 0's or -1's into the matrix. For this poset, it is defined as:

$\mu(y, x) = (-1)^k$ if y/x is a product of k distinct primes and 0 otherwise. This is the CLASSICAL Möbius function which comes from Number Theory and can be derived by using the Product Theorem on the partition of the set into chains of powers of primes: $p_1^0 < p_1^1 < p_1^2 < \ldots$, $p_2^0 < p_2^1 < p_2^2 < \ldots$ and so on and lattices combining the primes. You can see this here [Bender and Goldman, 1975]. Therefore we get our desired inverse matrix.

$$\begin{bmatrix} 1 & -1 & -1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

So for example if we want x_{10}, we multiply the 10th column with Y: 1(0) + 3(0) + 6(0) + 18(0) + 4(0) + 7(1) + 12(0) + 15(-1) + 28(-1) + 60(1) = 24.

So in conclusion, if we treat the functions g, f as vectors G, F, then the Möbius Inversion Theorem says:

$G = FZ \Rightarrow GM = F$, where Z and M are the Zeta and Möbius Matrices of the poset respectively.

And the proof of this is simply stating that $ZM = I$ (the identity matrix)!

8. Conclusion

In this expository paper, we talk about posets, build up to the Mobius inversion theorem, and then talk about its applications. We see how this theorem when applied to the right poset is able to reveal important results such as the fundamental theorem of calculus. There are many instances in which it gets tricky to calculate the Mobius function, such as in Convex Polytopes, Map colouring and more. I hope this paper peaked your interest enough to learn more about related topics.

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Acknowledgements

I want to thank Simon Rubenstein-Salzedo and Sawyer Dobson for helping and guiding me throughout the time of writing this paper.