van Der Waerden's Theorem

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Arithmetic Progression

Definition (Arithmetic Progression)

Arithmetic Progression - A group of $n+1$ numbers which are in the form $a, a+d, a+2d, a+3d \cdots a+(n-1)d,\ a+n(d)$ are said to be $n+1$ numbers in arithmetic progression.

Definition (Length and Step of Arithmetic Progression)

Using the above notation, the length of the arithmetic progression is $n+1$ and the step is d.

5 11 17 23 29

Game to explain the concept

- As seen we have nine marbles.
- We have 2 colors, red and blue.
- Please give me a coloring combination of these 9 marbles with red and blue.
- Your goal is to avoid an arithmetic progression of length 3 which is monochromatic (of the same color).

Result of the game for all cases

- Arbitrary values k and l are chosen.
- k is the number of colors.
- l is the length of arithmetic progression requires.
- van Der Waerden Theorem guarantees there is a large enough number of marbles n.

van Der Waerden's Theorem and van Der Waerden's Number

Theorem (van Der Waerden's Theorem)

Given any k and l we can always find a n such that if all the natural numbers from 1 to n were to be coloured with k colors, we can always find a group of l values which are in arithmetic progression and are monochromatic.

Definition (van Der Waerden's Number)

The smallest value n for which this is true, is the van der Waerden number $W(k, l)$.

Known Values

Determining the van Der Waerden number $W(k, l)$ is unknown for all but 7 pairs of k and l.

van Der Waerden's Theorem			
k / l	2 Colors	3 Colors	4 Colors
3	9	27	76
4	35	293	$\geq$ 1,048
5	178	$\geq$ 2173	$\geq$ 17,705
6	1,132	$\geq$ 11,191	$\geq$ 157,209
7	$\geq$ 3,703	$\geq$ 48,811	$\geq$ 2,284,751
8	$\geq$ 11,495	$\geq$ 238,400	$\geq$ 12,288,155

Does this have any bounds? The answer is yes.

Gowers Upper Bound

For any k and l the upper bound which was found by Gowers is: -

$$W(k,l) \leq 2^{2^{l^{2^{2^{(k+9)}}}}}.$$

For just the case of W(2, 3) we see that the upper bound will give us

$$W(2,3) \leq 2^{2^{2^{2^{4096}}}}.$$

while the actual value is 9.

Other Related Theorems

Some other related theorems are

- Hales–Jewett Theorem
- Rado's Theorem
- Szemerédi's Theorem
- Polynomial van Der Waerden Theorem
- Multidimensional van Der Waerden Theorem

They are all connected to

1. Ramsey Theory - "Complete disorder is impossible!"
2. Coloring
3. Additive Combinatorics

Pigeon Hole Principle

Definition (Pigeon Hole Principle)

If we have $n+1$ pigeons and n pigeon holes, then if we arrange them in any way, at least one of the pigeon holes must contain 2 pigeons.

Definition (Pigeon Hole Principle Generalisation)

If in the pigeon hole principle there are n pigeons and k pigeon holes then at least 1 of the k pigeon holes has more than $\lceil \mathrm{n/k} \rceil$ pigeons. The way in which we arrange the pigeons into pigeon holes does not matter.

Proof

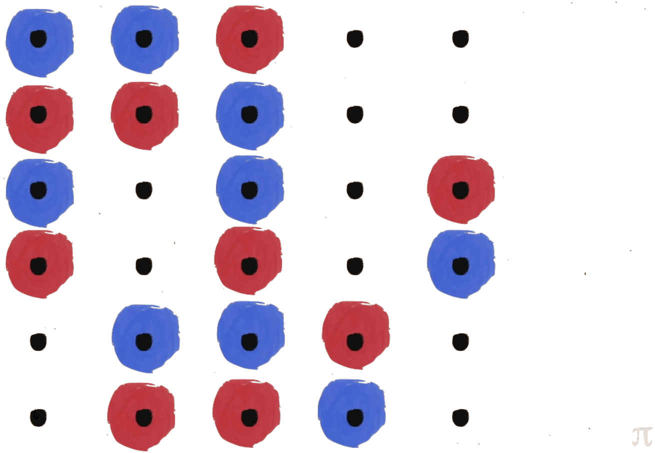
The key ideas of the proof of van Der Waerden's Theorem are as follows:

1. Finding van Der Waerden's number for smaller cases like $W(2,3)$, $W(3,3)$ and $W(2,4)$.
2. Constructing lemmas and understaningd the key ideas needed for the entire proof.
3. Building 3 equivalent blocks using Pigeon Hole Principle.
4. Understanding good blocks (example on next slide).

Proof of W(2, 3)

- Proof is extremely complex.
- We will just give a general idea of the case W(2, 3)
- Please read the expository paper I have written which goes through this in detail

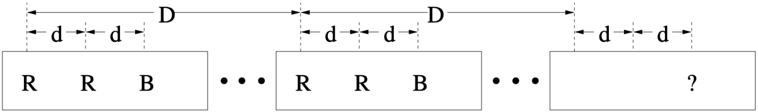
The Good Blocks for 2 colors



Different coloring and the Pigeon Hole Principle

1. Block W of length 5 which has a coloring u.
2. Each number has 2 coloring choices.
3. Total colorings u can be 32 (2^5).
4. Pigeon Hole Principle implies that in 33 blocks there must be 2 that are colored the same.

Key Idea and Lemma



[1]

[1]Van der Waerden's Theorem: Variants and "Applications" William Gasarch, Clyde Kruskal, Andy Parrish March 9, 2018

Q/A and Thank You

Seeing that I don't have enough time to answer questions during the presentation, please message on discord in the presentation-questions in Euler Circle server or direct message me on Samvar Shah#5307 regarding any questions, comments or suggestions.