Van der Waerden's Theorem

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Abstract

This paper will cover the fascinating topic of van der Waerden's Theorem which is about finding arithmetic progression of the same color when we color positive integers. We introduce the concepts of Ramsey theory, discuss the proof of van der Waerden's theorem, and mention some related results.

Introduction

We start off by asking the reader a question: Is it possible to avoid a monochromatic **arithmetic progression*** (i.e a group of numbers with a common difference) while coloring a set of positive integers of any **length*** (i.e.the number of values inside an arithemtic progression) and any **step*** (i.e.the common difference)? This question is answered by van der Waerden's Theorem. In this paper, we go through van der Waerden's Theorem, van der Waerden number(i.e. the smallest value for which the van der Waerden's Theorem works), the proof of van der Waerden's Theorem, and upper and lower bounds of van der Waerden's Theorem. Additionally, we will also review key supporting principles to understand van der Waerden's theorem, which include the Pigeon Hole Principle, some other related theorem and introduction to Ramsey Theorey (focusing on finding order in a combinatorial object). *Formally defined in Section 1.

Walkthrough

Section 1 goes through the notation that will be used in this paper. One of the key theorems of the paper is the Pigeon Hole Principle which we explain in Section 2. Section 3 briefly touches upon Ramsey Theory. In Section 4 we describe the theorem, its proof, known values and open problem. We give some references to other related theorems in 5, describe the Gowers and Berlekamp bounds in Section 6, 7 provide some other resources in Section 8, acknowledgements in Section 9 and finally the bibliography in Section 10.

1 Definitions

Definition 1.1. Arithmetic Progression - A group of n numbers $a, a+d, a+2d, a+3d \cdot a+(n-1)d,\ a+n(d)$ are said to be n numbers in arithmetic progression.[GKP12]

Definition 1.2. Length of Arithmetic Progression - The number of values in an arithmetic progression is said to be the length of the arithmetic progression. For example in the arithmetic progression $5, 9, 13, 17, 21, 25$, there are 6 values. This means that the length of this arithmetic progression is 6. [GKP12]

Definition 1.3. Step of Arithmetic Progression - The difference between any 2 successive values of an arithmetic progression is said to be the step of that arithmetic progression. For example in $5, 9, 13, 17, 21, 25$, the difference between 5 and 9 is 4. It is also 4 for every other subtraction of successive numbers. This means that the step of this arithmetic progression is 4. Note : The array of numbers has to be sorted in ascending or descending order for us to calculate the step in this method. [GKP12]

Definition 1.4. Monochromatic Subset - A subset in which each element has the same coloring is called a monochromatic subset.

Definition 1.5. Combinatorial Object - An object that can be put into one-to-one correspondence with a finite set of integers is known as a combinatorial object.

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2 The Essential Pigeon Hole Principle

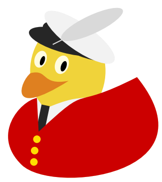
The Pigeon Hole Principle is actually a very simple theorem to understand.

Theorem 2.1. *Pigeon Hole Principle: Given $n+1$ pigeons and n pigeon holes, if we consider all the possible arrangements of pigeons into the available pigeon holes, we will find that in each arrangement atleast one of the pigeon holes contains 2 pigeons.*

Proof. If we had 2 pigeon holes and 3 pigeons and we wanted to arrange them in such a way that neither of the holes would have 2 pigeons we see that it is impossible as :-

1. We put pigeon 1 in hole 1 without the loss of generality. This means we have 2 more pigeons left but only 1 hole.
2. We put pigeon 2 in hole 2. Now we have pigeon 3 still left but no more vacant holes.
3. Thus pigeon 3 must be put in either hole 1 or hole 2 only as those are the only two holes we have, although they both already have one pigeon.

Now we consider the same principle in n, $n+1$ terms. When we arrange the first n pigeons into n *holes* we find that atleast 1 pigeon remains and that there are no vacant holes remaining. This means that atleast 1 hole will have more than 1 pigeon, ensuring that every pigeon is in a hole.



The generalisation of this principle:

Theorem 2.2. *Pigeon Hole Principle Generalisation: If n pigeons are placed into k pigeon holes where $n, k >= 1$ then atleast one of the k pigeon holes will contain at least $[n/k]$ pigeons where $[]$ stands for the ceiling function. [AZ1]*

3 Introduction to Ramsey Theory

Ramsey Theory is a branch of combinatorics which can be characterized by the brilliant statement made by Theodore S. Motzkin - "Complete disordrer is impossible!" or in other words, in whichever way we color a combinatorial object, there will certainly by a monochromatic subset present. What this means is that even if we try all possible random arrangements in a set, given a large enough value, we will find some order.One aspect of the Ramsey Theory is the Theorem of Friends and Strangers. Let us now consider the same. [ITR, GKP12]

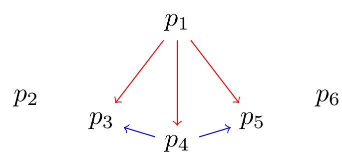
3.1 The Theorem of Friends and Strangers

Let us first consider a smaller portion of The Theorem of Friends and Strangers, the Dinner Party Problem.

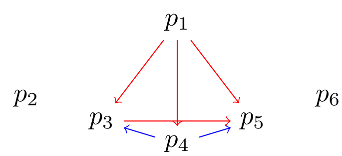
Theorem 3.1. *The Theorem of Friends and Strangers in a dinner party: Suppose a party has 6 people. Let us consider all the possible pairs of two of them. If they are meeting for the first time they are mutual strangers or if they have met before they are mutual friends. In any party if we observe all possible pairs of 2 persons, there must be either 3 pairs of people who know each other or do not know each other.*

This can be handily explained using graph coloring. Each of the people in the party are represented as a vertex. Let us connect each vertex to all friends in blue color and to all strangers in red color and then observe the graph. We will note that formation of a blue triangle would mean a group of 3 people knowing each other andthe formation of a red triangle would mean a group of 3 people not knowing each other.

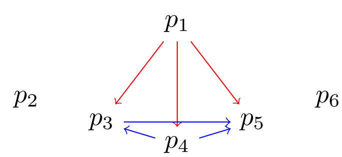
Proof. Let us consider each person as a vertex. Now using the Pigeon Hole Principle, from the possible 5 lines, atleast 3 must be of the same color. We construct this figure as shown below, without the loss of generality. Since P1-P3, P1-P4 and P1-P5 are red, this means P3-P4 and P4-P5 must be blue in order to prevent a red triangle. Now there are 2 possibilities, either P3-P5 is red or it is blue.



Case 1 - P3-P5 is red, this forms a red triangle P1-P3-P5.



Case 2 - P3-P5 is blue, this forms a blue triangle P3-P5-P4.



Therefore in every case there will be either a blue or a red triangle.

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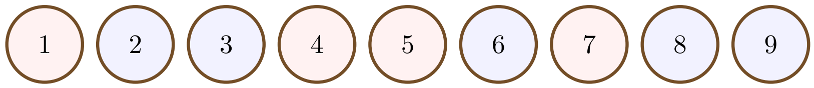
4 Van Der Waerden's Theorem

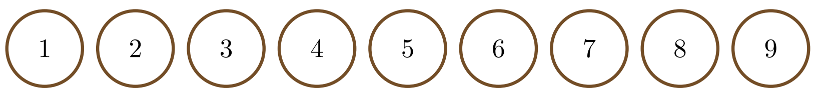
4.1 Introduction to van der Waerden's Theorem

Let us play a game to understand the basic concept of van der Waerden's Theorem. Consider that we have nine marbles which need to be colored using two colours, red and blue. The aim of the game is to avoid an arithmetic progression of length 3 in the same colour (either red or blue). Can you try some possible arrangements and check to see whether you can win this game? Suppose you try this case:



Clearly 1, 4, 7 is an arithmetic progression of length 3 which is monochromatic which is what you were trying to avoid. You could try agin.

Clearly 2, 5, 8 is an arithmetic progression of length 3 which is monochromatic and turns out to be exactly what we were trying to avoid. We could try another possible arrangement and yet another, again and again and again but here's a heads up. We could never win this game! Van der Waerden's Theorem guarantees the same!



For any coloring of these nine marbles with red and blue, we will always find a monochromatic subset which has an arithmetic progression with length 3. Note - In the next figure, 1/K means choosing any one color from k choices.

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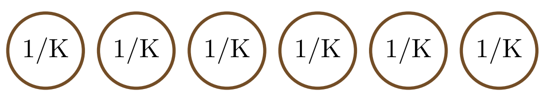


$\cdots$ and so on till n marbles.

In this case, we have k colors available and choose an arbitrary number l as the required length of arithmetic progression. We will certainly be able to find a value n which is the number of marbles, that satisfies the above conditions of finding a monochromatic arithmetic progression of length l. This is guaranteed by van der Waerden's Theorem.

Theorem 4.1. *Van Der Waerden's Theorem: We choose arbitary natural numbers k and l, where k is the number of colours we have at our disposal while l is the length of arithmetic progression required. Van der Waerden's Theorem states that we can always find a value n such that if all the natural numbers from 1 to n where to be coloured with k colors, we would find a group of l values which are in arithmetic progression and are monochramatic.*

4.2 Van der Waerden's Number

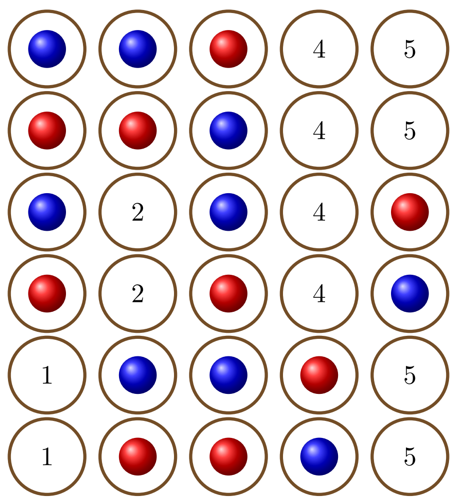
Definition 4.1. Van der Waerden's Number: The smallest value n, which is the number of positive integers that we color using k colors at our disposal, such that we are guaranteed a monochromatic subset with length of arithmetic progression being l, then the least possible integers required for this condition to be fulfilled, is the van der Waerden number W(k, l).

4.3 Open Problem

One of the most exciting parts about this topic is that it has an open problem. It is determining the van der Waerden number for all arbitary values of $W(k, l)$. The van der Waerden number is unkown for all but 7 arbitary pairs of k and l. However the proof of the van der Waerden Theorem gives us a nice upper bound for the van der Waerden number. Although there are better bounds known, let us go through the proof of van der Waerden's Theorem first.

4.4 Good Blocks

A good block is any configuration that occurs in a coloring where 2 numbers force the value of the third. [Num20]



4.5 Proof of van der Waerden's Theorem

4.5.1 Key Idea 1

Lemma 4.2. *The van der Waerden number for* 2 *colors, and length of arithmetic progrssion* 3*, is less than or equal to* 325*, that is* $W(2, 3) \geq 325$*.*

Proof. Let us now prove Lemma 4.2

1. For the sake of simplicity, let us assume that the number of blocks in our example is divisible by 5.

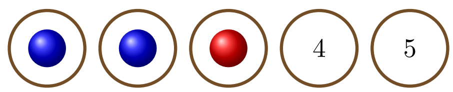
2. Seeing that each number can be colored in 2 different ways, red or blue, there are a total number of $2^5 = 32$ possible coloring combinations of one block.

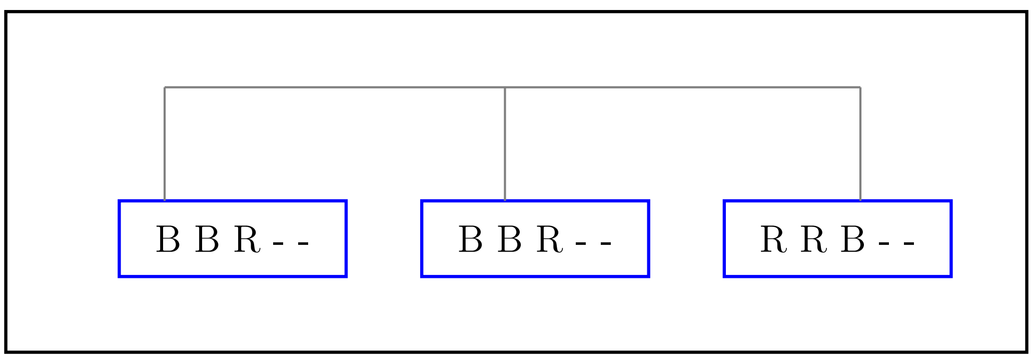
3. By the Pigeon Hole Principle, if we have 33 blocks, 2 of them must be equivalent.

4. By Number 3, there are 2 identical blocks and each of these must contain one of the good block configurations.

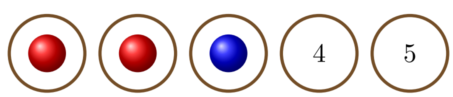
5. Now let us build the third block using case work.

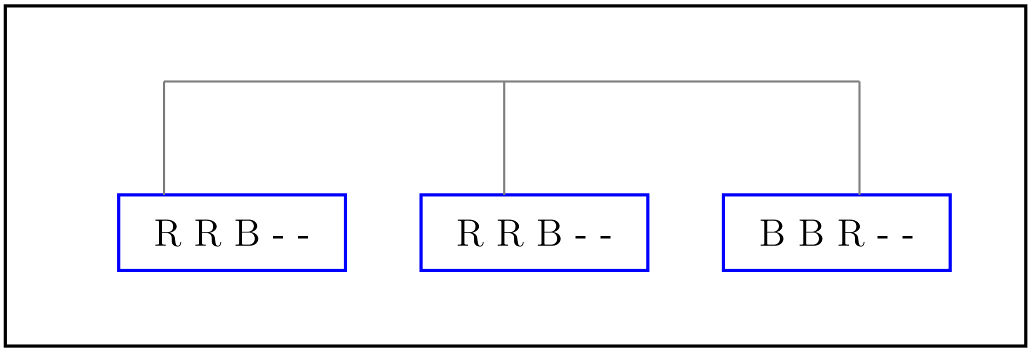
Case 1 - There is a clear arithmetic progression between the three blue values.



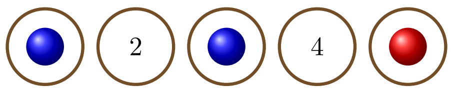


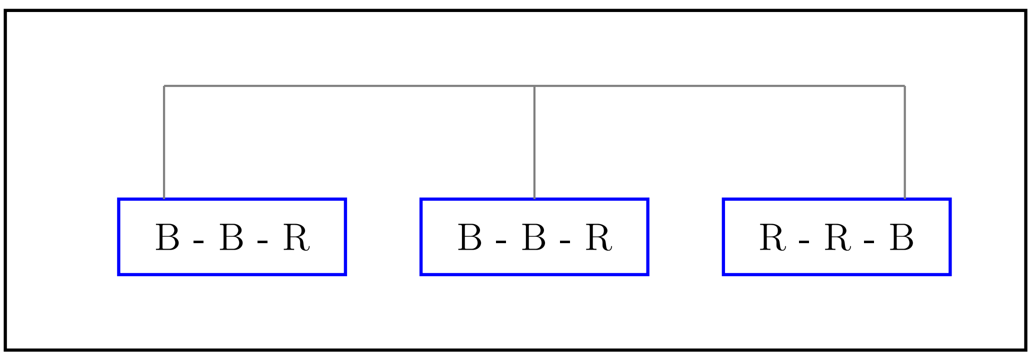
Case 2 - There is a clear arithmetic progression between the three red values.



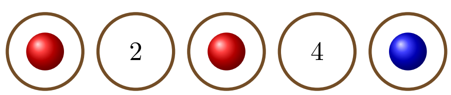


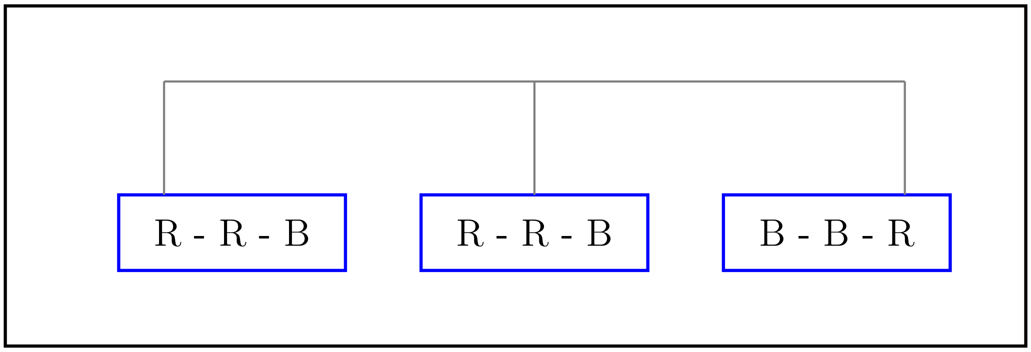
Case 3 - There is a clear arithmetic progression between the three red values.



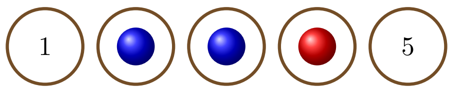


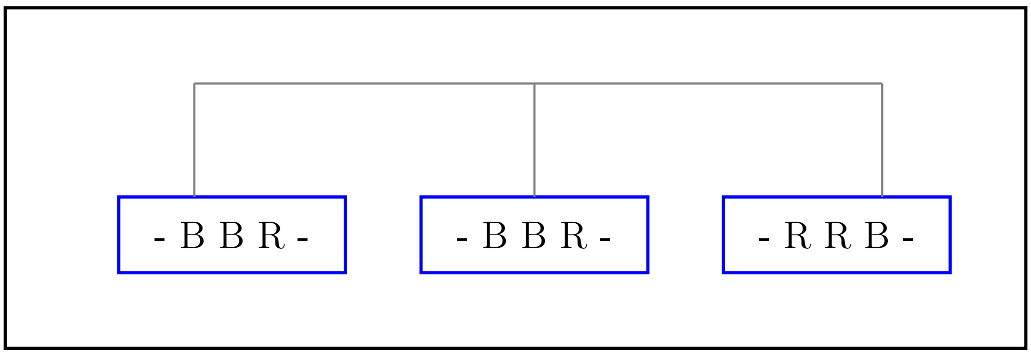
Case 4 - There is a clear arithmetic progression between the three red values.



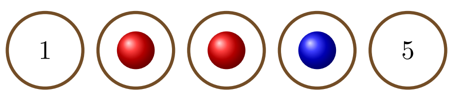


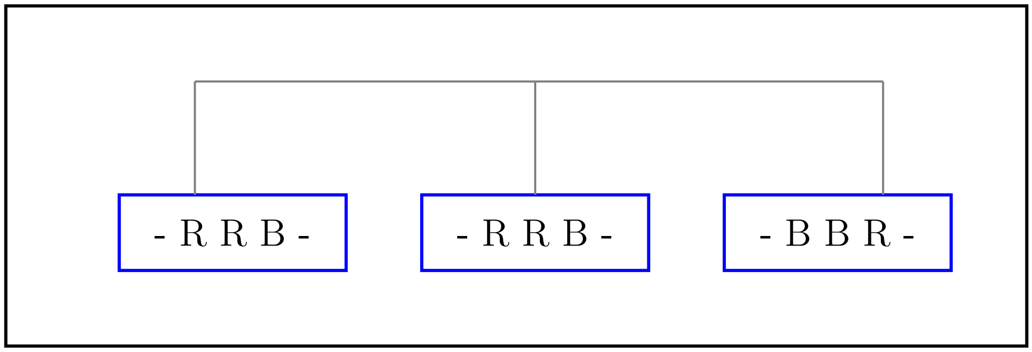
Case 5 - There is a clear arithmetic progression between the three red values.



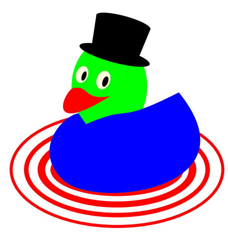


Case 6 - There is a clear arithmetic progression between the three red values.





In any group of 5 blocks, when we color using 2 colors, atleast 3 numbers must be of the same color, thus implying that atleast one of good blocks configuration exists. Since each configuration give us the ability to then build an arithmetic progression and since the coloring contains atleast one of these configurations, we have proved that the arithmetic progression occurs. The number of blocks = 32 (beginning) + 33 (to ensure equivalent blocks) * 5 = 65 * 5 =325. Thus $W(2,3) \geq 325$



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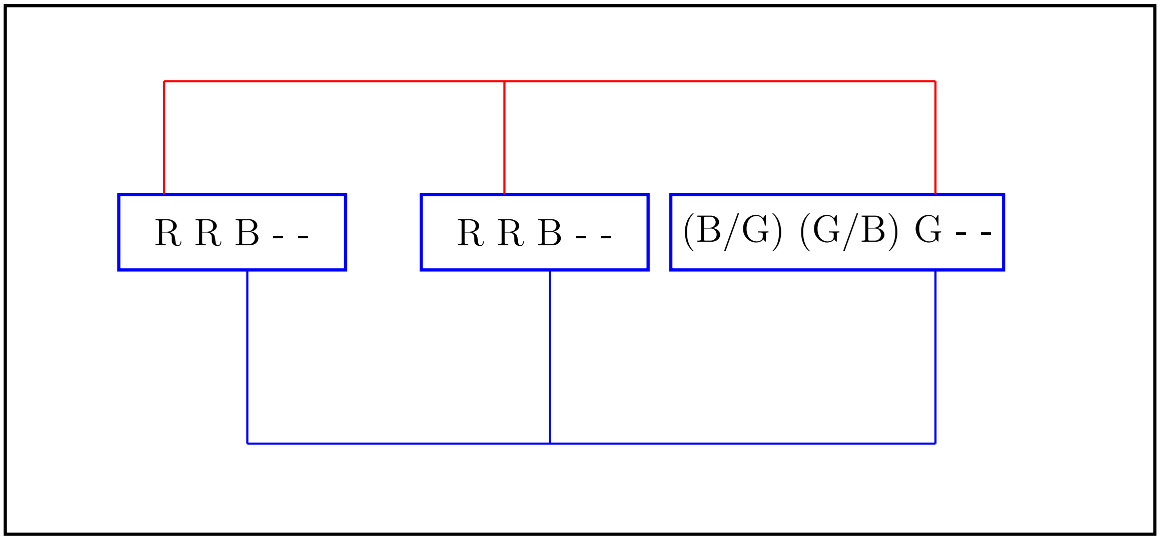
4.5.2 Key Idea 2

Lemma 4.3. *There exists a coloring U such that, for all 3-colorings of U one of the following must occur:*

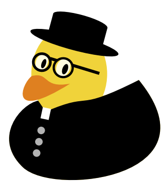
- *There exists a monochromatic arithmetic progression of length 3 or*
- *There exist two arithemtic progressions of length 3 such that*
  - *One is colored Red : Red : Green*
  - *One is colored Blue : Blue : Green They have the same third point.*
- *In other words, in both cases, the third number bears the same color.*

Proof. Let us now prove Lemma 4.3

The first thing we consider is that in any coloring, there is either an arithmetic progression present or no arithmetic progression exists. Point number 1 is the case where the momochromatic arithemtic progression of length 3 exists, while point 2 is a case where it does not exist.



We see that both the blue and red lines are looking at the exact same square which can neither be blue or red implying that it has to be green, proving the second part of Lemma 4.3.



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4.5.3 Key Idea 3

Lemma 4.4. *There exists a coloring U for $2W(k, 3)$*

- $f(a) = f(a + d) = f(a + 2d)$ *and*
- $f(a + 3d)$ *is defined, though we make no claims of its value.*

4.5.4 Final Lemma

Now that we have understood these key ideas and lemmas we can proceed to proving van der Waerden's Theorem, but before that we must define one last lemma.

Lemma 4.5. *If we have a coloring which is large enough, then:*

- *Either there will be a monochromatic k colored arithmetic progression or there will be an*

- *Arbitrarily large number of monochromatic k colored arithmetic progression, all of different colors.*

4.6 Complete Proof

We now prove van der Waerden's Theorem whose defintion we remind you is: - **van der Waerden's Theorem: We choose arbitary natural numbers k and l, where k is the number of colors we have at our disposal while l is the length of arithmetic progression required. Van der Waerden's Theorem then states that we can always find a value n such that if all the natural numbers from 1 to n where to be coloured with k colors, we would find a group of l values which are in arithmetic progression and are monochramatic.**

Proof. Let us consider the proof of van der Waerden's Theorem:

- *Induction Principle*: van der Waerden's Theorem can be proved using induction on l. That is we show that:
  - W(k, 1) exists
  - W(k, l) exists
- *Base Case*: $k = 1$ As noted above $W(k, 1)$ suffices. In fact, we note that $W(k, 2) = c + 1$ suffices.
- *Induction Step*: We assume that $W(2, k-1)$ suffices. Fix k. Consider what Lemma 4.5 says: It says, either there is a monochromatic l length arithmetic progression or there a k colored monochromatic $l-1$ length arithmetic progression which are all differently colored and a number a whose color differs from all of them. Since there are only k colors this cannot happen, so we must have a monochromatic l length arithmetic progression. Hence $W(k, l)$ exists. Note that the proof of $VDW(k, l)$ depends on $VDW(k-1, l\prime)$ where $l\prime$ is quite large. Formally the proof is an induction on the following order on N×N.

  $(1, 1) \prec (1, 2) \cdots \prec (2, 1) \prec (2, 2) \cdots \prec (3, 1) \prec (3, 2) \cdots \prec$

  This is an ω^2 ordering. It is well founded, so induction works.

□

4.7 Known Values

As mentioned previously, there are only 7 pairs of k and l for which we know van der Waerden's number.

Van der Waerden's Number			
k / l	2 Colors	3 Colors	4 Colors
3	9	27	76
4	35	293	$\geq$ 1,048
5	178	$\geq$ 2173	$\geq$ 17,705
6	1,132	$\geq$ 11,191	$\geq$ 157,209
7	$\geq$ 3,703	$\geq$ 48,811	$\geq$ 2,284,751
8	$\geq$ 11,495	$\geq$ 238,400	$\geq$ 12,288,155

5 Other Related Theorems

5.1 Hales Jewett Theorem

Information about the Hales Jewett Theorem can be found here: `https://en.wikipedia.org/wiki/Hales-Jewett_theorem`

5.2 Rado's Theorem

Information about Rado's Theorem can be found here: `https://en.wikipedia.org/wiki/Rado%27s_theorem_(Ramsey_theory)`

5.3 Szemerédi's theorem

Information about Szemerédi's Theorem can be found here: `https://en.wikipedia.org/wiki/Szemer%C3%A9di%27s_theorem`

6 Gowers' Upper Bound

For any k and l, the upper bound which was found by Gowers is: -

$$W(k,l) \leq 2^{2^{l^{2^{2^{(k+9)}}}}}.$$

We need to understand that the values it gives for the upper bound are very very large compared to the actual van der Waerden Number. For just the case of W(2, 3) we see that the upper bound the above formula will give us is:

$$W(2,3) \leq 2^{2^{2^{2^{4096}}}}.$$

But the actual value of the van der Waerden number for the above case is only 9.

One upper bound for the case of $W(2,3)$ is 325 which as we showed earlier, is also quite far away from 9 but the fact that the upper bound value exists, is quite important. It proves that the number is finite. This is the reason Gower's Upper Bound is so amazing as it guarantees the existence of van der Waerden's number for any natural number k and l.

7 Berlekamp's Lower Bound for Primes

For a prime number p, the 2-color van der Waerden number is bound below by:

$$p \cdot 2^p \leq W(2, p+1)$$

as proved by Berlekamp. The proof of this theorem can be found in the original paper of E.R.Berlekamp, where he proved the same.

8 Other Resources

Here are a few resources that would be highly interesting reads for the reader:

1. Three Pearls of Number Theory by A Y Khinchin
2. Van der Waerden's Theorem: Variants and "Applications" by William Gasarch, Clyde Kruskal, Andy Parrish
3. Introduction to Ramsey Theory: Lecture notes for undergraduate course.
4. Youtube video by field's medalist Timothy Gowers on the YouTube channel 'numberphile' regarding van der Waerden's Theorem (Part 1 and Part 2)
5. A Construction for Partitions Which Avoid Long Arithmetic Progressions by E.R.Berlekamp which proves the Berlekamp Lower Bound amongst other things.

9 Acknowledgements

The author would like to extend great thanks to Andrew Lin and Simon Rubinstein-Salzedo, for continous encouragement and invaluable mathematical advice in writing this paper.

10 Bibliography

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