The Axiom Of Choice (AC)

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The Finite Hat Paradox

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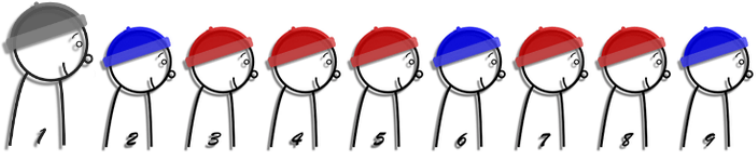
The Finite Hat Paradox

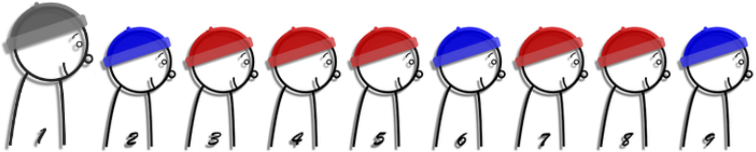
The Infinite Hat Paradox

Question : Each of you gets a hat to wear, which is either red or blue. You can see everyone in front of you, including the colors of their hats; you can't see your own hat, nor can you see anyone behind you. Starting at the back of the line, the host will ask each person to guess whether their own hat is black or white. You'll be able to hear the guesses, and whether they're right or wrong.

The Finite Hat Paradox

The Infinite Hat Paradox





Number of red hats : 5 (odd)
So they say “Red”.

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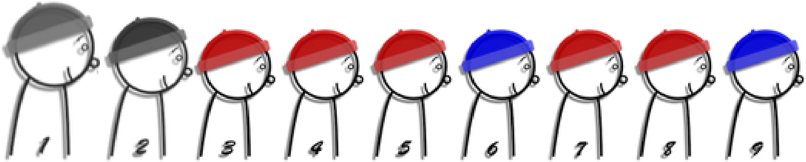

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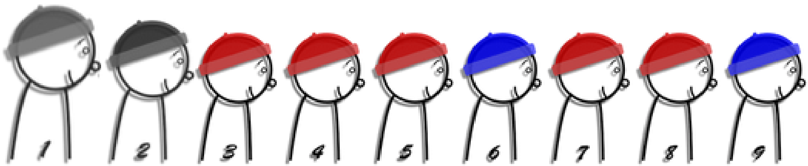


Xi said "Red"

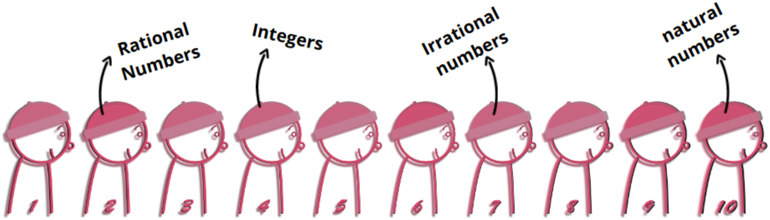
Number of red hats : 5 (still odd)

So they say "Blue".

The Infinite Hat Paradox

Question : There is an infinite line of people. All of them are wearing hats, the hats have random real numbers on them. Assume there is a first person in the line, so the line is infinite only in one direction, and one cannot hear the answers of the people preceding them.

What should you do to get as many correct guesses as possible?



The Infinite Hat Paradox

The Infinite Hat Paradox

- Let A be the set of people wearing the hat, S be the set of all possible numbers sequences on said hats (in our case $\mathbb{R}^{\mathbb{N}}$), $\prec$ be a binary, transitive relation on A, and X be the set of all $A \to S$.
- For agents s and a, $s \prec a$ means that a can see the number on agent s's hat, and $f \sim g$ means that scenarios f and g are indistinguishable to agent a. Given a scenario f for the numbers on the hats, $[f]_a$ is the set of scenarios consistent with what a can see.
- An equivalence relation for this problem would look like $\mathbb{N} \sim \mathbb{R}^{\mathbb{N}}$ such that they're different in only finitely many places.

- Now let $\mathbb{O} = \{[\alpha]_a \mid \alpha \in X\, and\, a \in A\}$.
- A well-ordering $\preceq$ on X such that $\mu : \mathbb{O} \to X$ and $\mu([f]_a)$ is the $\preceq$ least element of $[f]_a$.[1]
- We would fix an element $\alpha \in X$ which we consider to be the true scenario.

[1]This strategy may be viewed as a formalization of Occam's Razor if we interpret $f \prec g$ as f is "simpler" than g in some sense.

The Infinite Hat Paradox

Those “finitely many places” wherein the equivalence relations differed are the number of people who may get the answer wrong; the other infinitely many are sure to get it right.

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2 The Axiom of Choice

The Axiom of Choice

The Axiom of Choice

If A is a collection of nonempty mutually disjoint sets, then we can find a set C that has exactly one element in common with every set from A.
It is may also be defined as: for any indexed collection of sets, there exists a choice function.

$$f : X \to \bigcup_{i \in X} S_i$$

such that $f(i) \in S_i$ for all $i \in X$.

Consequences

The Axiom of Choice

1. Let $C = (\forall x)(\forall y)([x, y] \in \mathbb{R}^+)$ such that each interval has a finite length. Then we can define the choice function $f(C)$ to be the midpoint of the interval $[x, y]$.

Consequences

The Axiom of Choice

1. Let $C = (\forall x)(\forall y)([x, y] \in \mathbb{R}^+)$ such that each interval has a finite length. Then we can define the choice function $f(C)$ to be the midpoint of the interval $[x, y]$.
2. In Bertrand Russell's words "To choose one sock from each of infinitely many pairs of socks requires the Axiom of Choice, but for shoes the Axiom is not needed."

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3 The Well-Ordering Theorem & Zorn’s Lemma

Equivalent Statements to AC

The Well-Ordering Theorem & Zorn's Lemma

Theorem (Well-Ordering Theorem)

Given any set S, there exists a well-order $\prec$ on S.

Equivalent Statements to AC

The Well-Ordering Theorem & Zorn's Lemma

Theorem (Well-Ordering Theorem)

Given any set S, there exists a well-order $\prec$ on S.

Theorem (Zorn's Lemma)

Every non-empty partially ordered set in which every totally ordered subset has an upper bound contains at least one maximal element.

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4 Consequences of AC

The Banach-Tarski Paradox

Consequences of AC

Theorem (Banach-Tarski Paradox)

The unit ball B^3 is equidecomposable with two copies of itself.

Simply, it states that one can take a 3-dimensional ball, cut it into a finite number of disjoint sets consisting of infinitesimally small pieces (points), perform simple rotations on said sets, and then put them back together to yield two copies of the original ball.

The Banach-Tarski Paradox

Consequences of AC

- Equidecomposibility : An An analogue of polygonal dissections.
- Equidecomposability as action of a group G on a set X.

Definition

If G acts on X, we say that $A, B \subseteq X$ are G-equidecomposable if they can be partitioned into the same finite number of pieces, which can be matched such that each pair of corresponding pieces A_i, B_i are related by the action of some $g_i \in G \mid B_i = g_i A_i = \{g_i a | a \in A_i\}$.

- Free group with two generators can be realized as a group of rotations of our sphere.
- B^3 is equidecomposable with $B^3 \setminus \{\beta\}$ where β is the center of the ball. $S^2 \setminus D$ can be partitioned into two sets, both being equidecomposable with $S^2 \setminus D$.
- $\{\beta\}$ can be duplicated using a simple circle trick.

Consider a circle that passes through β and is contained within B^3. Let ρ be a 1 rad rotation of the circle, then the points $\beta, \rho\beta, \rho^2\beta, \rho^3\beta, \dots$ are distinct and we get the same set except β upon applying ρ on them.
This yields an analogous to the 2-piece decomposition of the B^3 with $B^3 \setminus \beta$: one piece being $\{\rho\beta, \rho^2\beta, \rho^3\beta, \dots\}$,[2] and the being the rest of the ball to itself.

[2] from $\{\beta, \rho\beta, \rho^2\beta, \rho^3\beta, \dots\}$ under ρ

The Banach-Tarski Paradox: Some Intuitive Examples

Consequences of AC

The proof made use of a lot of group theory, and might've been hard to follow. So here are some examples to help you understand what is happening a little better.
Example 1:

$$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$$

The Banach-Tarski Paradox: Some Intuitive Examples

Consequences of AC

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Example 1:

$$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$$

Upon subtracting 1 from each of the elements we get

$$\mathbb{N} - 1 = \{0, 1, 2, 3, 4, 5, \ldots\}$$

The Banach-Tarski Paradox: Some Intuitive Examples

Consequences of AC

Example 2:
Let C be the unit circle in $\mathbb{R}^2$, $r = (0, 1)$, a line segment on the x-axis, be a radius of C, ρ be a counterclockwise rotation by $\frac{1}{5}$ rads around the origin, so $p(r)$ is another radius of C.
So

$$\mathcal{D} = \bigcup_{n=0}^{\infty} = \rho^n(r).$$

The Banach-Tarski Paradox: Some Intuitive Examples

Consequences of AC

Example 2:
Let C be the unit circle in $\mathbb{R}^2$, $r = (0,1)$, a line segment on the x-axis, be a radius of C, ρ be a counterclockwise rotation by $\frac{1}{5}$ rads around the origin, so $p(r)$ is another radius of C.
So

$$\mathcal{D} = \bigcup_{n=0}^{\infty} = \rho^n(r).$$

We now rotate $\mathcal{D}$ clockwise by $\frac{1}{5}$ rads. And because $\rho^n(r) \neq r$ for any n,

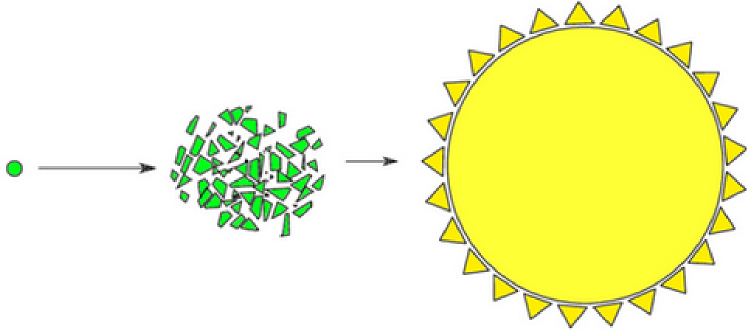
$$\rho^{n-1}(r) \neq \rho^{-1}(r)$$

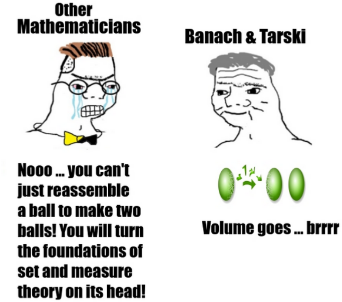
$$C \bigcup \rho^{-}1(\mathcal{D}) = C \bigcap \mathcal{D} \bigcap \rho^{-}1(r)$$

The Banach-Tarski Paradox

Consequences of AC

By extension of the Banach-Tarski Theorem it can be shown that one can start with any bounded set with a nonempty interior and reassemble it into any other such set of any volume, so that one could, in principle, begin with a pea and end up with a ball as large as the Sun.

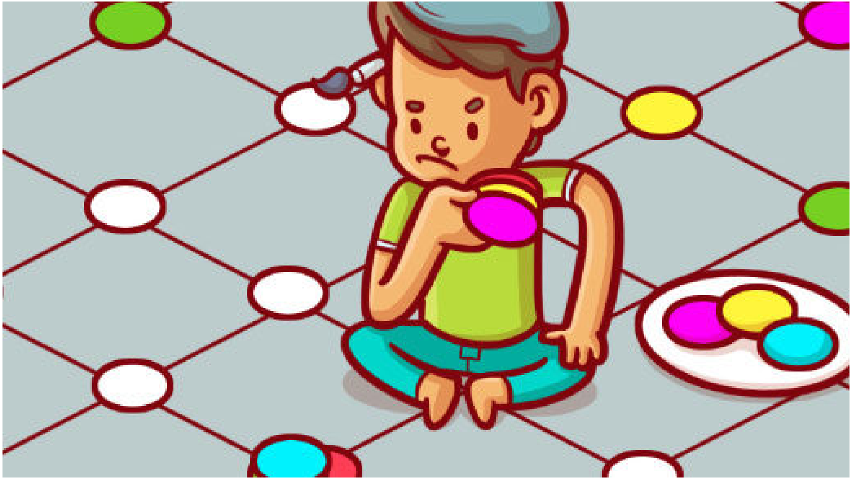




Because the pieces are infinitesimally small *their volume is degenerate.* Or, to be more precise they are not Lebesgue measurable.

The De Bruijn–Erdős Theorem

Consequences of AC



The De Bruijn–Erdős Theorem

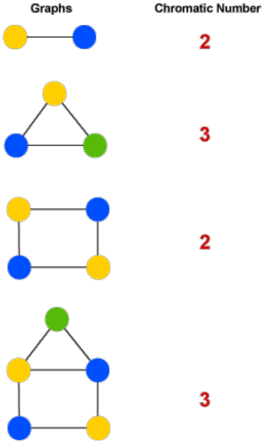
Consequences of AC

Theorem (De Bruijn–Erdős Theorem)

If the chromatic numbers of the finite subgraphs of a graph G have a finite maximum value c, then the chromatic number of G itself is exactly c. On the other hand, if there is no finite upper bound on the chromatic numbers of the finite subgraphs of G, then the chromatic number of G itself must be infinite.

The De Bruijn–Erdős Theorem

Consequences of AC



Chromatic number of graphs having different number of nodes

The De Bruijn–Erdős Theorem

Consequences of AC

What would be the chromatic number of the unit distance graph?

The De Bruijn–Erdős Theorem

Consequences of AC

What would be the chromatic number of the unit distance graph?

Seven. Well, not exactly. We know it cannot be four, so the answer is somewhere between five and seven.

The De Bruijn–Erdős Theorem

Consequences of AC

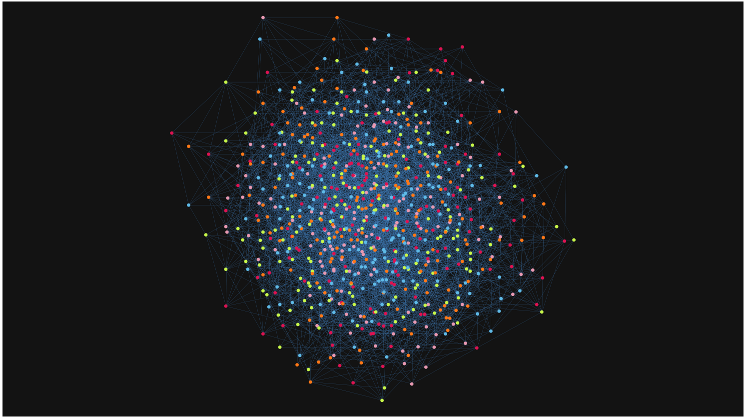


Figure: This 826-vertex unit distance graph requires at least five colors to ensure its proper coloring.

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5 Negation of AC

Looking at the world without Choice

Negation of AC

The Cartesian product of two non-empty sets is empty. Let X be a collection of non-empty sets, and P be a set in X. Intuitively, we know that

$$\forall X[\phi \notin X \Rightarrow \exists f : X \to \bigcup X \ \forall P \in X(f(P)) \in P)]$$

But if we consider the negation of AC, there exists no choice function such as f. Because of which a Cartesian product cannot be constructed, and is empty.

Looking at the world without Choice

Negation of AC

Some other consequences of $ZF\neg C$ include :

- In some model, the real numbers are a countable union of countable sets. [3]
- In all models of $ZF\neg C$ there is a vector space with no basis.
- In all models of $ZF\neg C$, the generalized continuum hypothesis does not hold.

[3] This does not imply that the real numbers are countable: To show that a countable union of countable sets is itself countable requires the Axiom of countable choice.

Thank You!

Your feedback will be highly appreciated!