Combinatorial species

Nicole Shen
shennicole2004@gmail.com

Euler Circle

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Definitions

Definition (Combinatorial Species)

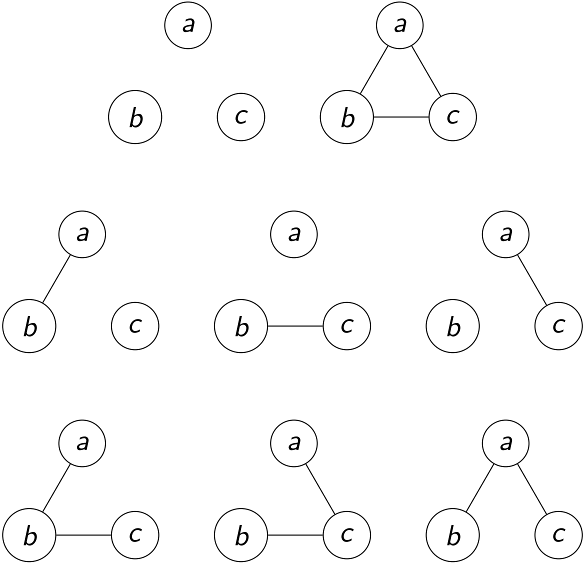
A *combinatorial species* F is a function that sends a finite set U of labels to a finite set of structures $F[U]$.

Example

L is the species of linear orders built from the set $\{a, b, c\}$:

$$\left\{ \begin{array}{ccc} a < b < c, & b < c < a, & c < a < b, \\ a < c < b, & b < a < c, & c < b < a \end{array} \right\}.$$

Species Example



Equivalence

Definition (Equivalence)

Two species F and G are *equivalent* (written as $F \approx G$) if they are naturally isomorphic.

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Definition (Equipotence)

Two species F and G are *equipotent* (denoted as $F \equiv G$) if and only if $|F[U]| = |G[U]|$ for all finite sets U.

Generating Functions

Definition (Exponential Generating Function)

For a species F, the associated egf is given by

$$F(x) = \sum_{n=0}^{\infty} |F[n]| \frac{x^n}{n!}$$

where $|F[n]|$ is the number of labeled F-structures with size n.

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Example

$$L(x) = \sum_{n=0}^{\infty} n! \frac{x^n}{n!} = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

Operations - Addition

Definition (Addition)

$$(F + G)[U] = F[U] \sqcup G[U],$$

the disjoint union of $F[U]$ and $G[U]$.

Operations - Multiplication

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$$(F \cdot G)[U] = \bigsqcup_{U} F[A] \cdot G[B].$$

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Example

Let E be the species of sets. The $E \cdot E$ species partitions the set of labels into two distinguishable parts. This is equivalent to the species of subsets, P.

$$P(x) = (E \cdot E)(x) = E(x) \cdot E(x) = e^{2x} = \sum_{n=0}^{\infty} \frac{2^n \cdot x^n}{n!}.$$

Thus, a set with n elements has 2^n subsets.

Operations - Composition

Definition (Composition (aka substitution))

$$(F \circ G)[U] = \bigsqcup_{U = \sqcup_{i \leq n} B_i} F[n] \cdot \prod_{i \leq n} G[B_i].$$

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Example

The EGF of the species of cyclic orderings C is $\sum_{n=0}^{\infty} \frac{x^n}{n}$. A permutation is a set of cycles, so

$$S(x) = E(C(x)) = e^{C(x)} = \frac{1}{1-x}.$$

This means

$$C(x) = \ln\left(\frac{1}{1-x}\right) = -\ln(1-x).$$

$$\rightarrow -\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^n}{n}.$$

Operations - Differentiation

Definition (Differentiation)

The derivative of the species F is given by

$$F'[U] = F[U \sqcup \{*\}]$$

where $\{*\}$ is a distinguished point.

Why? $|F'[n]| = |F[n+1]|$!

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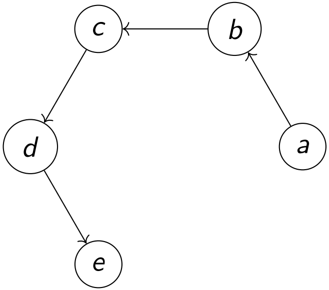

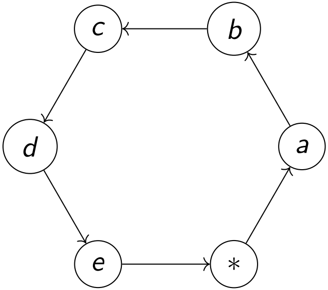

Operations - Pointing

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Let F be a species. Pointing is used to select one of the n elements of the underlying set U as "special."

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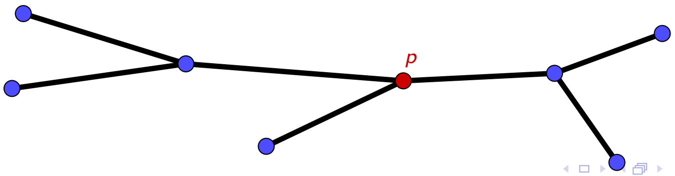
Example

Let a be the species of trees and A be the species of rooted trees.

$$\rightarrow A = a^{\bullet}$$



Cayley's Theorem

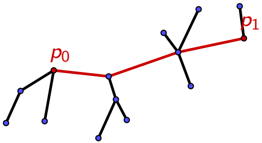
Lemma

Vertebrates can be seen as linear orders of rooted trees.

Proof.

A vertebrate is a tree bipointed by two vertices: the tail vertex (p_0) and the head vertex (p_1). Along the spine, each vertex is the root of a rooted tree.

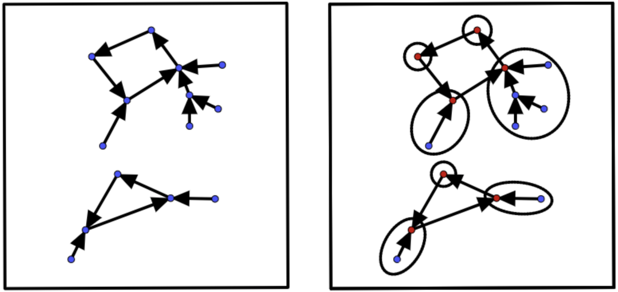
$$V(x) = L(A(x)) = \frac{1}{1 - A(x)}.$$



Cayley's Theorem

Definition (Endofunction)

An *endofunction* is a function whose codomain is equal to its domain.



$$End(x) = S(A(x)) = \frac{1}{1 - A(x)}.$$

Cayley's Theorem

Theorem

There are n^{n-2} labeled trees on n labeled vertices.

Proof.

$V = a^{\bullet\bullet}$, so $|V[n]| = n^2 \cdot |a[n]|$.

$$V(x) = End(x) = \frac{1}{1 - A(x)}.$$

Since $|End[n]| = n^n$,

$$n^2 \cdot |a[n]| = n^n.$$

Thus,

$$|a[n]| = n^{n-2}.$$

Other Applications

- proving the Lagrange Inversion Theorem!
- providing the foundations of Polya's Enumeration Theory
- and more!