COMBINATORIAL SPECIES

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Abstract. We explore the essence of combinatorial species. Generating functions, operations on species, and classic applications are discussed. We also introduce group theory fundamentals, along with cycle index polynomials and ultimately, Polya's Enumeration Theorem.

Introduction

This ideas in combinatorial species are largely based on André Joyal's paper from 1981, which used species to give new proofs for Cayley's Theorem and Lagrange Inversion Formula, as well as reframe Polya's Enumeration Theorem. This paper covers these major ideas and provides the fundamentals necessary to understand these ideas.

Acknowledgements

The author would like to thank Dr. Simon Rubinstein-Salzedo and Euler Circle for making this paper possible and Kishan Jani for helpful guidance along the way.

1. Definitions and Basics

Definition 1.1 (Combinatorial Species). A *combinatorial species* F is a function that sends a finite set U (of labels) to a finite set of structures $F[U]$. For any bijection $\sigma : U \to V$, a species F constructs a function $F[\sigma] : F[U] \to F[V]$, which must satisfy these functor properties:

(1) For the identity map $Id_U : U \to U$, $F[Id_U] = Id_{F[U]}$.
(2) For all bijections $\sigma : U \to V$ and $\tau : V \to W$, $F[\sigma \circ \tau] = F[\sigma] \circ F[\tau]$.

Intuitively, this means that F produces all possible structures built on U.

Notation. The elements of $F[U]$ are called "F-structures on U, and we can say that U underlies $F[U]$. Furthermore, $F[\sigma]$ is known as "the transport of F along σ" or "the relabelling of F-structures along σ

Example. L is the species of linear orders built from the set $\{a, b, c\}$. Refer to Figure 1.

$$\{a, b, c\} \underset{L}{\rightarrow} \left\{ \begin{array}{lll} a < b < c, & b < c < a, & c < a < b, \\ a < c < b, & b < a < c, & c < b < a \end{array} \right\}.$$

Figure 1. the species of linear orders on $\{a, b, c\}$

Date: July 11, 2022.

Example. G is the species of graphs with underlying set of vertices $\{a, b, c\}$. We can transport the labels of the graphs. For example, the permutation $\sigma = (123)$ would map the graphs as shown in Figure 2.

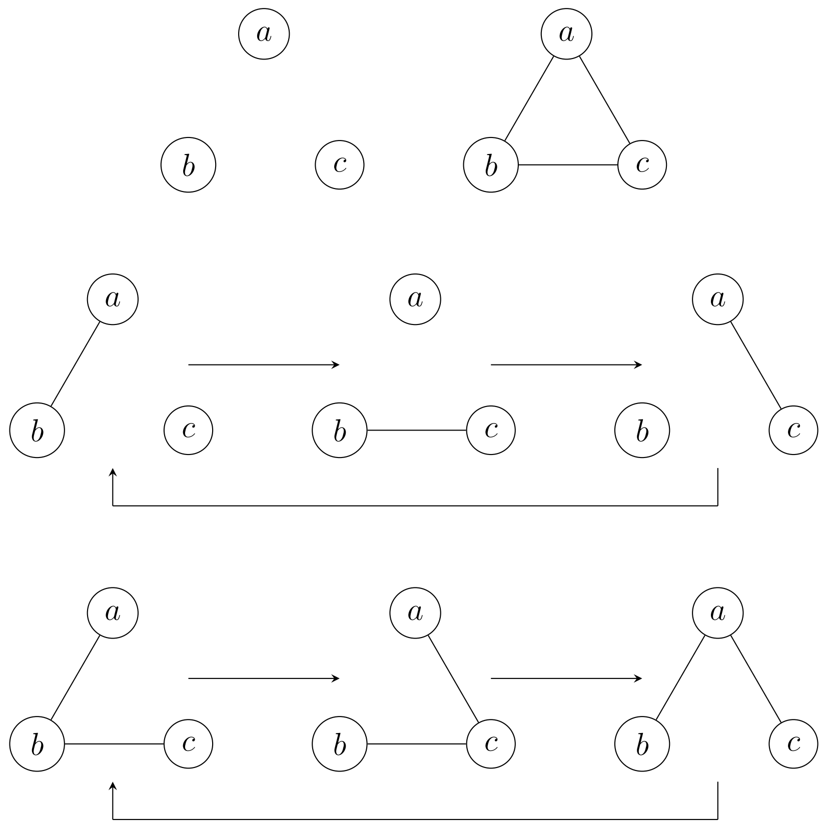

Figure 2. Application of the permutation (123) on the structures of G

Definition 1.2 (Equivalence)**.** Two species F and G are *equivalent* (written as $F \approx G$) if they are naturally isomorphic. They have the same fundamental internal structure, and are invariant under relabelling.

Definition 1.3 (Equipotence)**.** Two species F and G are *equipotent* (denoted as $F \equiv G$) if and only if $|F[U]| = |G[U]|$ for all finite sets U.

Essentially, equipotence is a weaker form of equivalence and is useful in counting labeled structures.

Example. Let's example the species L of linear orders and the species S of permutations. From a set U with n elements, we have $|L[U]| = |S[U]| = n!$. Thus,

$$L \equiv S.$$

However, they are not equivalent, or naturally isomorphic. We observe that any two linear orderings are equal up to relabelling, while this is not true for permutations since the relabelling could not map cycles that have the different sizes.

2. Generating Functions

We can associate three generating functions with a species: an exponential generating function (EGF), ordinary generating function (OGF), and the cycle index series.

Definition 2.1 (Exponential Generating Function)**.** For a species F, the associated EGF is given by

$$F(x) = \sum_{n=0}^{\infty} |F[n]| \frac{x^n}{n!}$$

where $|F[n]|$ is the number of labeled F-structures with size n.

Definition 2.2 (Ordinary Generating Function)**.** For a species F, the associated OGF is given by

$$\tilde{F}(x) = \sum_{n=0}^{\infty} |F[n]| x^n.$$

OGF's are useful in counting unlabeled structures in $F[U]$. They are used to enumerate objects up to their automorphisms.

For now, we will focus on the egf, which are generally most convenient to use when dealing with combinatorial enumeration problems involving labeled objects. The cycle index series will be covered in Section 7.

Example. Let L be the species of linear orders. As discussed earlier, $|L[n]| = n!$. Then,

$$L(x) = \sum_{n=0}^{\infty} n! \frac{x^n}{n!} = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

The species S has the same egf since $|S[n]| = n!$ as well.

In essence, the egf $F(x)$ is a way to algebraically represent the species F. Consider the quote is "A generating function is a clothesline on which we hang up a sequence of numbers for display" [Wilf, 1990]. We will now work through a more interesting example that demonstrates this.

Example. We will find the generating function of the Fibonacci sequence. Its recursive relationship is

$$F_n = F_{n-1} + F_{n-2}$$

with $F_0 = 0, F_1 = 1$.

Let's define the generating function to be

$$F(x) = \sum_{n=0}^{\infty} f_n x^n.$$

Then, taking advantage of the "shifting" nature of multiplying x, we can write the following equations.

$$\begin{aligned} F(x) &= 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + \cdots \\ xF(x) &= x + x^2 + 2x^3 + 3x^4 + 5x^5 + \cdots \\ x^2F(x) &= x^2 + x^3 + 2x^4 + 3x^5 + \cdots . \end{aligned}$$

Observe that

$$F(x) = x^2F(x) + xF(x) + 1,$$

which can be rewritten as

$$F(x) = \frac{1}{1 - x - x^2}.$$

We can now try to find f_n.

From here, we can factor the denominator and apply partial fraction decomposition to get

$$F(x) = \frac{a}{1 - cx} + \frac{b}{1 - dx}$$

where

$$a = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right),$$

$$b = -\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right),$$

$$c = \frac{1+\sqrt{5}}{2},$$

$$d = \frac{1-\sqrt{5}}{2}.$$

Since the Taylor series expansion for $\frac{1}{1-kx}$ is

$$\frac{1}{1-kx} = 1 + kx + k^2x^2 + k^3x^3 \cdots,$$

we can expand and manipulate the expression for $F(x)$ to get

$$f_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right).$$

We have shown Binet's formula using generating functions!

3. Operations

We can construct species from other species using operations. Operations are also useful for offering interpretations in combinatorial proofs.

Definition 3.1 (Addition)**.** The sum of species F and G is the species such that

$$(F + G)[U] = F[U] \sqcup G[U],$$

the disjoint union of $F[U]$ and $G[U]$.

Conveniently, operations on species carry over nicely to generating functions: $(F+G)(x) = F(x) + G(x)$.

Example. Let E_o be the species of odd sets and E_e be the species of even sets. Then,

$$E_o(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \sinh(x)$$

and

$$E_e(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = \cosh(x).$$

Then, the species of sets E has egf

$$E(x) = \sinh(x) + \cosh(x) = e^x.$$

Definition 3.2 (Multiplication)**.** The product of species F and G is formed by partitioning the set U of labels: $U = A \sqcup B$.

$$(F \cdot G)[U] = \bigsqcup_U F[A] \cdot G[B].$$

Luckily again, we have $(F \cdot G)(x) = F(x) \cdot G(x)$.

We will now examine some examples showing how these operations could be useful combinatorially.

Example. Algebraic identities can be rewritten in using species.

$$\frac{1}{1-x} = 1 + x \cdot \frac{1}{1-x}.$$

X is the singleton species, which consists of one object. Its generating function is $X(x) = x$. Since we also have $L(x) = \frac{1}{1-x}$,

$$L \approx 1 + X \cdot L.$$

Combinatorially, this means a linear order (L) is either empty (1) or (+) can be written as a single label (X) followed by ($\cdot$) another linear order (L).

Example. Let E be the species of sets. Then by definition, the $E \cdot E$ species partitions the set of labels into two distinguishable parts. This is equivalent to the species of subsets, P.

$$P(x) = (E \cdot E)(x) = E(x) \cdot E(x) = e^{2x} = \sum_{n=0}^{\infty} \frac{2^n \cdot x^n}{n!}.$$

This shows that a set with n elements has 2^n subsets.

Definition 3.3 (Composition (aka substitution))**.** Let F and G be species such that G has no structures on the empty set ($G[\emptyset] = \emptyset$). We construct $(F \circ G)[U]$ by partitioning the underlying set U into n nonempty sets: $U = B_1 \sqcup B_2 \sqcup ... \sqcup B_n$. Then, we put a G-structure on each of these sets, and finally put a F-structure on the set of G-structures.

$$(F \circ G)[U] = \bigsqcup_{U = \sqcup_{i \leq n} B_i} F[n] \cdot \prod_{i \leq n} G[B_i].$$

Again, $(F \circ G)(x) = F(G(x))$.

One common composition is with the species E of finite sets. Combinatorially, we can build the members of $(E \circ F)[U]$ by partitioning U into nonempty sets and putting an F structure on each of these sets. Composing the generating functions yields

$$(E \circ F)(x) = e^{F(x)}.$$

Here is an interesting proof using composition.

Example. Let C be the species of cyclic orderings. We find the EGF to be

$$C(x) = \sum_{n=0}^{\infty}(n-1)! \cdot \frac{x^n}{n!} = \sum_{n=0}^{\infty}\frac{x^n}{n}.$$

A permutation is a set of cycles, so we can write

$$S(x) = E(C(x)) = e^{C(x)} = \frac{1}{1-x}.$$

This means

$$C(x) = \ln\left(\frac{1}{1-x}\right) = -\ln(1-x).$$

We have proven that

$$-\ln(1-x) = \sum_{n=0}^{\infty}\frac{x^n}{n}$$

without using calculus! However, since this is only a formal power series, we can't actually say anything about whether it converges.

Definition 3.4 (Differentiation)**.** The derivative of the species F is given by

$$F'[U] = F[U \sqcup \{*\}]$$

where $\{*\}$ is a distinguished point.

In order to construct a species F' such that $F'(x) = \frac{d}{dx}F(x)$, we need $|F'[n]| = |F[n+1]|$ by the definition of the exponential generating function. This is why taking the derivative is equivalent to adding a $*$ to the set (while ensuring that it is invariant under isomorphisms).

Example. From Figure 3, we can see that the species L of linear orders is the derivative of the species C of cyclic orderings since we can imagine "unfolding" the cyclic ordering at $*$ to get a linear order.

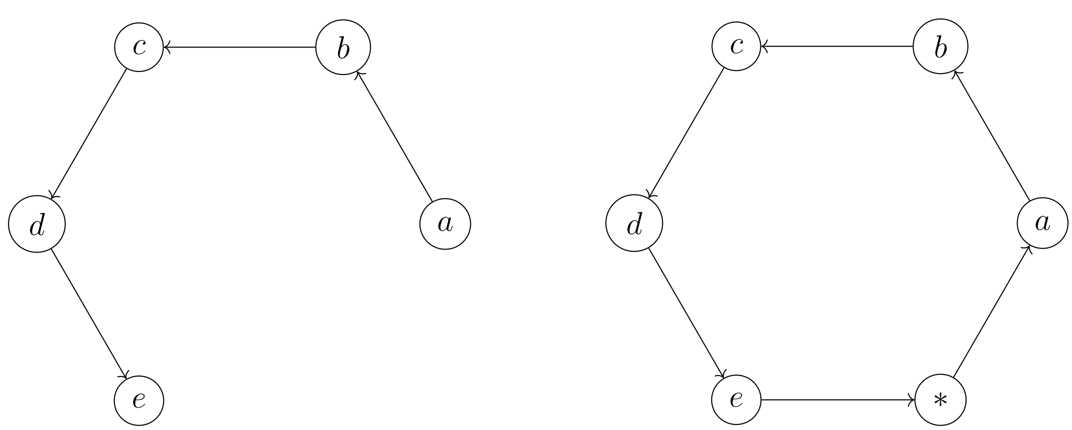

Figure 3. $C' = L$

We can use this to find the generating series of C to be

$$C(x) = \int_0^x \frac{1}{1-x} dx = \ln(\frac{1}{1-x}).$$

Definition 3.5 (Pointing)**.** Let F be a species. Pointing is used to select one of the n elements of the underlying set U as "special."

$$F^\bullet[U] = F[U] \times U.$$

The similarities with differentiation gives us

$$F^\bullet \cong X \cdot F'$$

where X is the singleton species.

Example. Let a be the species of trees and A be the species of rooted trees. Then A is $a^\bullet$ since it has one "special" node, as shown in Figure 4.

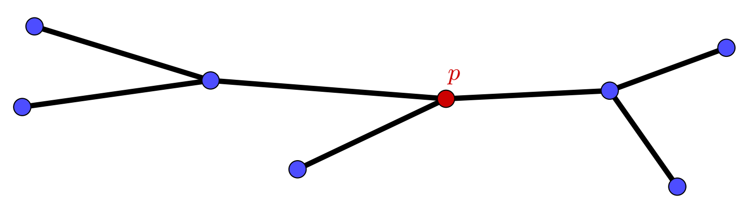

Figure 4. $A = a^\bullet$

4. Brief Applications

Now, before we dive deeper into proving theorems, let's solve a few problems using combinatorial species and generating functions.

Question 4.1. *What is the probabililty of a random permutation of* $[2n]$ *having all even cycles?*

Solution. Let C_2 be the speciess of even cyclic permutations and P_2 be the species of permutations with all even cycles.

First, note that permutations with all even cycles is the set of even cyclic permutations. In other words,

$$P_2 = E(C_2).$$

The generating function of the species C of cycles is $C(x) = \sum_{n=0}^{\infty} \frac{x^n}{n}$, so

$$\begin{aligned} C_2(x) &= \sum_{n=0}^{\infty} \frac{x^{2n}}{2n} \\ &= \frac{1}{2} \sum_{n=0}^{\infty} \frac{(x^2)^n}{n} \\ &= \frac{1}{2} \ln\left(\frac{1}{1-x^2}\right) \\ &= \ln\left(\frac{1}{\sqrt{1-x^2}}\right). \end{aligned}$$

This means

$$P_2 = e^{\ln\left(\frac{1}{\sqrt{1-x^2}}\right)} = \frac{1}{\sqrt{1-x^2}}.$$

We then expand with binomial theorem:

$$\begin{aligned} P_2(x) &= (1-x^2)^{-1/2} \\ &= \sum_{n=0}^{\infty} \binom{-1/2}{n} (-1)^n x^{2n} \\ &= \sum_{n=0}^{\infty} \binom{-1/2}{n} (-1)^n (2n)! \frac{x^{2n}}{(2n)!}. \end{aligned}$$

We can then extract the number of permutations with all even cycles from the EGF. Since the total number of permutations of $[2n]$ is $(2n)!$, the desired probability is

$$\frac{|P_2[2n]|}{(2n)} = (-1)^n \binom{-1/2}{n}.$$

It is easy to expand and manipulate this expression algebraically to get

$$\frac{|P_2[2n]|}{(2n)} = \frac{1}{2^{2n}} \binom{2n}{n},$$

which is our answer.

Question 4.2. *Count the alternating permutations of* $1, 2, \ldots, n$*, where an alternating permutation is one that can be written as*

$$a_1 < a_2 > a_3 < a_4 \cdots < a_{n-1} > a_n.$$

Solution. Let F be the species of alternating permutations. First, consider choosing the alternating permutation from $1, 2, \ldots, n+1$ instead; this is the original problem with the additional element $n+1$. This is equivalent to taking the derivative of F.

We will now construct F. $n+1$ clearly can not be the first or last element of the permutation, so we can split the permutation into three parts: $n+1$, the permutation before $n+1$, and the permutation after $n+1$. We can easily confirm that both permutations are alternating: the last element of the "before" permutation is a greater than the previous, and the first element of the "after" permutation is less than the second element.

Thus, we can write

$$F'(x) = F(x) \cdot F(x) + 1.$$

(We add one for the $n = 0$ case.)

Combined with $F(0) = 0$, we can solve for $F(x)$:

$$F(x) = \tan(x).$$

5. Cayley's Theorem

Now, we move on to proving Cayley's Theorem!

Lemma 5.1. *Vertebrates can be seen as linear orders of rooted trees.*

Proof. A vertebrae is a tree bipointed by two vertices: the tail vertex (p_0) and the head vertex (p_1), as shown in Figure 5. Along the shortest path between p_0 and p_1 (the spine), each vertex is the root of a rooted tree. Thus, we have $V = L(A)$ where V is the species of vertebrates.

$$V(x) = L(A(x)) = \frac{1}{1 - A(x)}.$$

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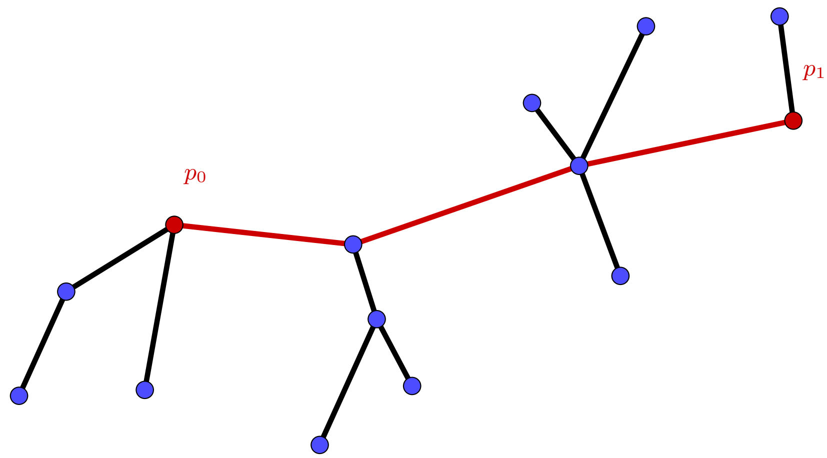

Figure 5. $V = a^{\bullet\bullet}$

Definition 5.2 (Endofunction)**.** An *endofunction* is a function whose codomain is equal to its domain.

Definition 5.3 (R-enriched)**.** The endofunction $\phi : E \to E$ is R-*enriched* if each of its fibers $\phi^{-1}\{x\}$ for $x \in E$ is equipped by the R species. (Don't forget the empty fibers.)

Lemma 5.4. *Let* End *be the species of endofunctions, S be the species of permutations, and A be species of rooted trees.*

$$\text{End} = S(A).$$

Proof. Let ϕ be an endofunction $\phi \in \text{End}[E]$. A point $x \in E$ is periodic if there exists $n \geq 1$ such that $\phi^n(x) = x$. Note that the periodic points are permuted by ϕ. Define the function $v(x)$ to be the first periodic point of the sequence $x, \phi(x), \phi^2(x), \ldots$. This essentially uncovers the roots of the rooted trees. Each fiber $v^{-1}\{x\}$ is equipped with a rooted tree structure with root x.

Working in the other direction, a permutation of rooted trees can easily be used to construct the corresponding endofunction.

Refer to Figure 6 for a visual explanation.

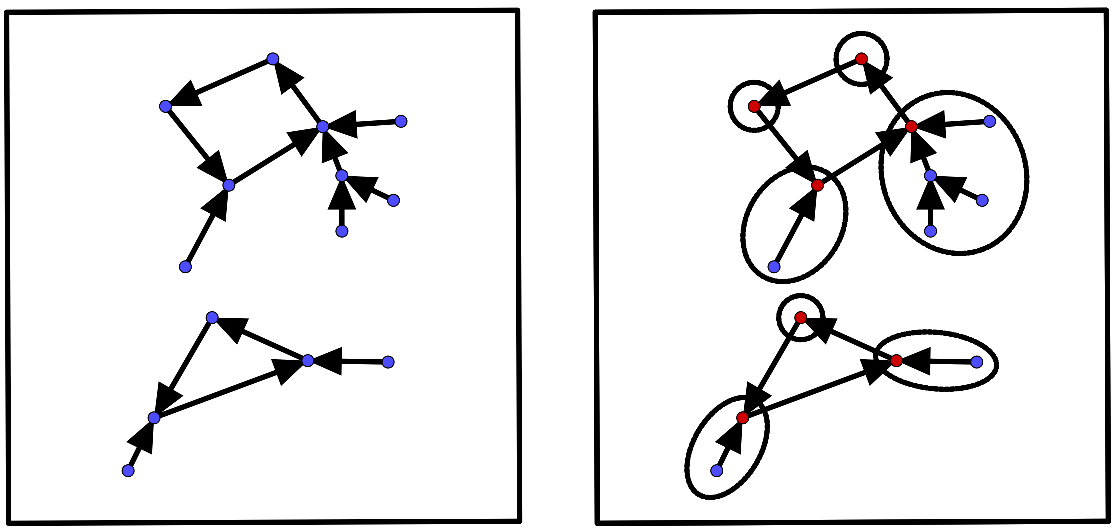

Figure 6. $\text{End} = S(A)$

Thus, $\text{End} = S \circ A$ and

$$\text{End}(x) = S(A(x)) = \frac{1}{1 - A(x)}.$$

■

Putting this all together, the proof of Cayley's theorem with species is straightforward.

Theorem 5.5. *There are n^{n-2} labeled trees on n labeled vertices.*

Proof. With two pointed vertices, the species V of vertebrates is $V = a^{\bullet\bullet}$ where a is the species of all trees.

$$|V[n]| = n^2 \cdot |a[n]|.$$

Combining lemma 4.1 with lemma 4.3, we see

$$V(x) = \text{End}(x) = \frac{1}{1 - A(x)}.$$

Since $|\text{End}[n]| = n^n$,

$$n^2 \cdot |a[n]| = n^n.$$

Thus, as desired,

$$|a[n]| = n^{n-2}.$$

■

6. Langrange Inversion Formula

The Lagrange Inversion Formula gives the Taylor series expansion of the inverse of an analytic function. In this section, we will walk through Joyal's species proof to reach one of the fundamental results of enumerative combinatorics. Aside from the obvious application of inverting functions, the Lagrange Inversion Formula can also be used to calculate coefficients of generating functions.

The most commonly written form of the formula is as follows:

Theorem 6.1 (Lagrange Inversion Formula)**.** *Suppose that $f(x)$ is analytic at point x and $f'(x) \neq 0$. Then, we can invert the equation and solve for the coefficients of the inverse formal power series $f^{-1}(x)$:*

$$f^{-1}(x) = \sum_{n \geq 1} \left(\frac{d}{dt}\right)^{n-1} \left(\frac{t}{f(t)}\right)^n \bigg|_{t=0} \frac{x^n}{n!}.$$

We can manipulate this formula in order to make it more suitable for the species proof.

Let $A(x)$ be the inverse of $f(x)$ and let $R(x) = \frac{x}{f(x)}$. By plugging in $A(x)$, we get the functional equation

$$A(x) = xR(A(x)).$$

Then,

$$A(x) = \sum_{n \geq 1} a_n \frac{x^n}{n!}$$

where

$$a_n = \left(\frac{d}{dt}\right)^{n-1} (R(t))^n \bigg|_{t=0}.$$

Thus, for any formal power series $F(x)$, we have

$$F(A(x)) = \sum_{n \geq 1} b_n \frac{x^n}{n!}$$

where $b_0 = F(0)$ and for $n \geq 1$,

$$b_n = \left(\frac{d}{dt}\right)^{n-1} F'(t)(R(t))^n \bigg|_{t=0}.$$

This is equivalent to showing

$$[x^n]F(A(x)) = \frac{1}{n}[t^{n-1}]F'(t)R^n(t).$$

We will now prove the species version of this:

Theorem 6.2 (Lagrange Inversion Formula)**.** *Let R and F be species and A_R be the species of R-enriched rooted trees. For $n \geq 1$, we have:*

$$F(A_R)[n] \equiv F'R^n[n-1]$$

(where $\equiv$ denotes equipotence).

First, we introduce some lemmas.

Lemma 6.3 (Labelle (1981))**.** *The species $C_R = X \cdot R'(A_R)$ coincides with that of R-enriched contractions, where a contraction is an endofunction that ultimately becomes constant.*

Proof. Let $\phi : E \to E$ be an R-enriched contraction with point of convergence $x_0 \in E$. (Every $x \in E$ satisfies $\phi^n(x) = x_0$ when n is sufficiently large.) For E $\{x_0\}$, there is a partition of A_R structures where each class is equipped with an R'-structure, as shown in Figure 7 from Joyal's paper. ■

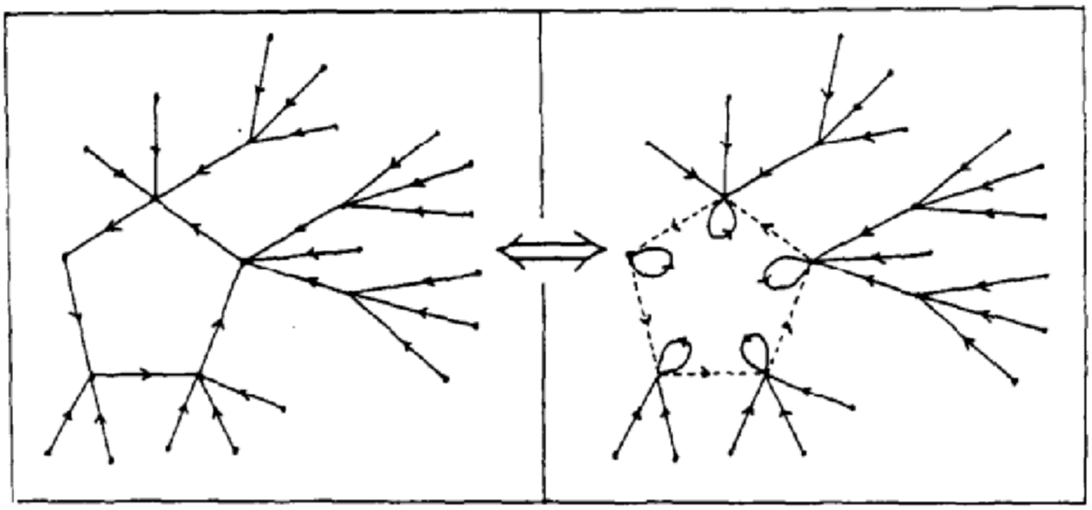

Figure 7

Lemma 6.4 (Labelle)**.** *The species $S(C_R)$ coincides with the species D_R of R-enriched endofunctions.*

$$S(C_R) = D_R.$$

Proof. Recall the proof of $D = S(A)$ (Lemma 4.4). Replacing the rooted trees with contractions, we get a bijection between the fiber of the endofunction and the fiber of the contraction. We can transform the R-structures according to this bijection to show there exists a bijection between R-enriched endofunctions (D_R) and the permutation of R-enriched contractions ($S(C_R)$).

■

Lemma 6.5 (Labelle, Repartitioning Lemma)**.** *For any species G, a $G(A_R)D_R$-structure on a set E can be interpreted as partitioning E into $E = E_1 \sqcup E_2$ such that E_1 is equipped with a G-structure γ and E_2 is equipped with an R-enriched.function $\lambda : E_2 \to E$.*

Proof. The product of species $G(A_R)$ and D_R can be formed by partitioning $E = F_1 \sqcup F_2$:

$$G(A_R) \cdot D_R = \bigsqcup_U (G(A_R))[F_1] \cdot D_R[F_2].$$

In other words, we have a forest of R-enriched rooted trees on F_1, with each root equipped with a G-structure, and an endofunction on F_2. These define an R-enriched function: $\lambda : E_2 \to E$. Conversely, given E_1, E_2, γ, and λ, we can find F_1, F_2, and the structures on them.

For example, F_1 is the set of all points in E that get transformed into E_1 by λ. This becomes clearer with Joyal's diagram in Figure 8.

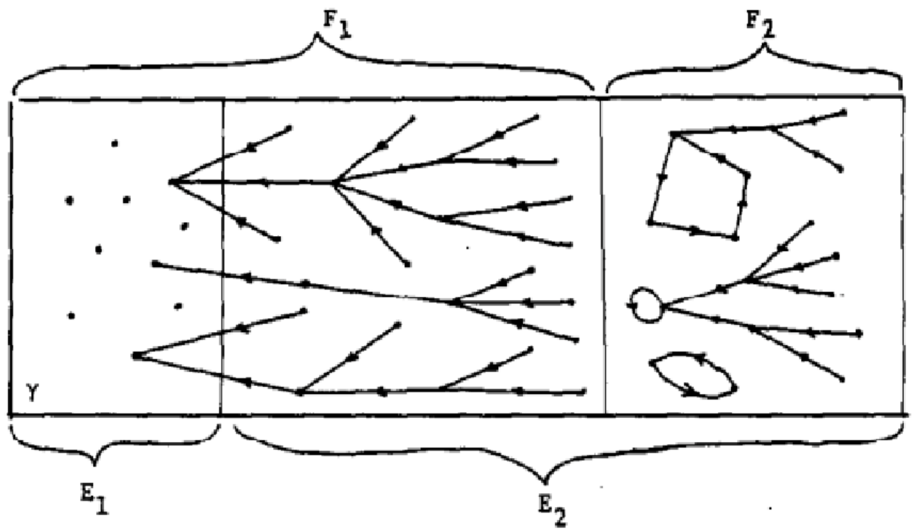

Figure 8

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Proof. First, note that the identity $A_R = X \cdot R(A_R)$ holds by construction. Differentiating both sides yields

$$A_R' = R(A_R) + X \cdot R'(A_R)A_R'.$$

Using $C_R = X \cdot R'(A_R)$, we get

$$A_R' = R(A_R) + C_R A_R'.$$

Here, we can solve for A_R':

$$A'R = R(A_R)\frac{1}{1 - C_R}.$$

and substitute $S(C_R) = \frac{1}{1-C_R}$.

We then take the derivative of $F(A_R)$ and substitute out expression for A_R' using the results from Lemma 6.3 and Lemma 6.4, which also offer combinatorial interpretations of $S(C_R)$:

$$\begin{aligned} F(A_R)' &= F'(A_R)A_R' \\ &= F'(A_R)R(A_R)\frac{1}{1 - C_R} \\ &= F'(A_R)R(A_R)S(C_R) \\ &= F'(A_R)R(A_R)D_R. \end{aligned}$$

Finally, with the Lemma 6.5, we can compute the cardinality of $(G(A_R)D_R)[n]$. When we let $E = [n]$, we get

$$G(A_R)D_R)[n] = (G \cdot R^n)[n].$$

Returning to the original statement, we finish by solving for $F(A_R)[n]$:

$$\begin{aligned} F(A_R)[n] &= F'(A_R)[n-1] \\ &= F'(A_R)R(A_R)D_R[n-1] \\ &= (F'R)R^{n-1}[n-1] \\ &= F'R^n[n-1]. \end{aligned}$$

■

The Lagrange Inversion Theorem has many applications. An interesting one is explicitly solving quintic polynomials of the form

$$x^5 - x - a = 0,$$

resulting in the infinite series

$$x = -\sum_{k\geq 0} \binom{5k}{k} \frac{a^{4k+1}}{4k+1}.$$

We can also reprove the now-familiar Cayely's Theorem using the Lagrange Inversion Theorem.

Theorem 6.6. *There are n^{n-2} labeled trees on n labeled vertices.*

Proof. This is equivalent to showing that there are n^{n-1} rooted trees by the pointing principle. (Refer back to Figure 3.)

Let A be the species of rooted trees and F be the species of rooted forests. Since a forest is a collection of trees, this isomorphism applies:

$$F \approx E \circ A.$$

To construct a rooted tree on a set of vertices, we can pick a vertex and let its neighbors be the roots of the rooted trees in a rooted forest. Then, we have

$$A \approx X \cdot F \approx X \cdot (E \circ A)$$

where X is the singleton species.

The generating function equivalent is

$$A(x) = xe^{A(x)},$$

which we can manipulate to get

$$A(x)e^{-A(x)} = x.$$

This means $A(x)$ is the compositional inverse of xe^{-x}. We now find the coefficients of the EGF with the Lagrange Inversion Theorem.

$$\begin{aligned} \frac{|A[n]|}{n!} &= \frac{1}{n}(x^{n-1})e^{nx} \\ &= \frac{1}{n}(x^{n-1})\sum_{k=0}^{\infty}\frac{(nu)^k}{k!} \\ &= \frac{1}{n}\cdot\frac{n^{n-1}}{(n-1)!} \\ &= \frac{n^{n-1}}{n!}. \end{aligned}$$

This means $|A[n]| = n^{n-1}$.

■

7. Cycle Index Series

In this subsection, we introduce cycle index series, which encodes both the labeled and unlabeled enumerative information of the species it is associated with. They are also important for Polya's Enumeration Theory, which will be discussed later.

Definition 7.1 (Cycle Index Series)**.** For a species F, the cycle index series (denoted Z_F is a formal power series in an infinite number of variables $x_1, x_2, x_3 \dots$.

$$Z_F(x_1, x_2, x_3 \dots) = \sum_{n\geq 0} \frac{1}{n!} \left(\sum_{\sigma \in S_n} |\operatorname{Fix} F[\sigma]| x_1^{\sigma_1} x_2^{\sigma_2} x_3^{\sigma_3} \cdots \right),$$

where $S_n = S[n]$, $\operatorname{Fix} F[\sigma]$ is the set of objects in the class F fixed under σ, and σ_i is the number of cycles with length i in the permutation σ.

Example. To get a better understanding of how cycle index series work, we will briefly examine the cycle index series of the species G of graphs.

When $n = 3$, we can make eight graphs. (Refer back to Figure 1.) The permutation $\sigma = (123)$ fixes two structures, so its term is $2x_3$, while the permutation $\sigma = (12)(3)$ permutes two pairs of structures, making its term $4x_1x_2$.

Example. To calculate the cycle index series of the species S of permutations, we first have

$$|\operatorname{Fix} F[\sigma]| = 1^{\sigma_1}\sigma_1! 2^{\sigma_2}\sigma_2! \cdots$$

by definition. Then, a permutation of summands and quick algebra yields

$$\begin{aligned} Z_S(x_1, x_2, x_3, \dots) &= \sum_{n_1 + 2n_2 + \cdots < \infty} (1^{n_1} n_1! 2^{n_2} n_2! \cdots) \frac{x_1^{n_1} x_2^{n_2} x_3^{n_3} \cdots}{1^{n_1} n_1! 2^{n_2} n_2! \cdots} \\ &= \left(\sum_{n_1} \sum_{n_2} \cdots \right) (x_1^{n_1} x_2^{n_2} x_3^{n_3} \cdots) \\ &= \frac{1}{(1-x_1)(1-x_2)\cdots}. \end{aligned}$$

We can also easily relate the cycle index series to EGFs and OGFs:

Theorem 7.2. *For any species F, we have the following:*

(1) $F(x) = Z_F(x, 0, 0, \dots)$
(2) $\tilde{F}(x) = Z_F(x, x^2, x^3, \dots)$.

Proof. It is relatively simple to prove (1). Substituting $(x_1, x_2, x_3 \dots) = (x, 0, 0, \dots)$ gives

$$Z_F(x, 0, 0, \dots) = \sum_{n\geq 0} \frac{1}{n!} \left(\sum_{\sigma \in S_n} |\operatorname{Fix} F[\sigma]| x_1^{\sigma_1} 0^{\sigma_2} 0^{\sigma_3} \cdots \right).$$

Note that unless $\sigma_1 = n$ and $\sigma_i = 0$ for $i \geq 2$, $x_1^{\sigma_1} 0^{\sigma_2} 0^{\sigma_3} \cdots = 0$ for each fixed value of n. Thus, only the identity permutation $\sigma = Id_n$ contribute to the sum.

$$\begin{aligned} Z_F(x, 0, 0, \dots) &= \sum_{n \geq 0} \frac{1}{n!} |\operatorname{Fix} F[Id_n]| x^n \\ &= \sum_{n \geq 0} \frac{1}{n!} f_n x^n \\ &= F(x). \end{aligned}$$

To prove (2), we use Burnside's lemma (whose proof will be explained in the next section), which states for any group G acting on a finite set X, the number of equivalence classes ω is

$$\omega = \frac{1}{|G|} \sum_{g \in G} |X^g|$$

where X^g is the subset of X fixed by action by g.

Let S_n act on $F[1, 2, \dots, n]$ to match the terms in Burnside's Lemma. Substituting, we get

$$Z_F(x, 0, 0, \dots) = \sum_{n \geq 0} \omega_n x^n$$

where ω_n denotes the number of objects of size n up to equivalence, as desired. ■

Thus, cycle index series can be seen as a generalization of EGFs and OGFs and contain more information than either.

8. Group Theory Fundamentals

Ideas in Polya's Enumeration Theory are polished using species and generating functions, so we will briefly explore them in the following sections.

First, we will momentarily edge away from combinatorial species and introduce some basic group theory definitions and lemmas.

Definition 8.1 (Group)**.** A group is a set G equipped with an operation $*$ satisfying associativity, identity, and inverse.

- Associativity: For all $a, b, c \in G$, $(a * b) * c = a * (b * c)$.
- Identity: There exists an identity element in G such that for all a, $e * a = a * e = a$.
- Inverse: For all $a \in G$, there exists $a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = e$.

Example. $\mathbb{Z}$ is a group under addition, with identity element 0. The inverse of a is $-a$.

We denote H being a subgroup of G as $H \leq G$. H has the same operation as G and all of its elements are contained in G.

One important group is the symmetric group S_n. Its elements are all permutations of $[n]$ and its operation is composition. For every finite set U, there is an implied symmetric group $Sym(U)$.

Groups also act on sets.

Definition 8.2 (Group Action)**.** For a group G and a set X, the left group action is a function $\phi : G \times X \to X$ that satisfies the following properties:

- Left Identity: For the identity element $e \in G$, all $x \in X$, satisfy $\phi(e, x) = x$.

- Left Compatibility: For all $g, h \in G$, and all $x \in X$, $\phi(gh, x) = \phi(g, \phi(h, x))$.

A right group action is defined similarly as a function $\phi : G \times X \to X$ that satisfies the following properties:

- Right Identity: For the identity element $e \in G$, all $x \in X$, satisfy $\phi(e, x) = x$.
- Right Compatibility: For all $g, h \in G$, and all $x \in X$, $\phi(x, gh) = \phi(\phi(x, g), h)$.

Unless otherwise specified, all group actions will be left group actions. We denote $\phi(g, x) = gx$ if ϕ is a left group action, and $\phi(x, g) = xg$ if ϕ is a right group action.

We have a few more definitions to introduce.

Definition 8.3. For the group action ϕ of group G on set X:

- For some $x \in X$, the *orbit* of x (written as $Orb_G(x)$ or O_x) is the set of all elements $y \in X$ such that $g \in G$ which satisfies $y = gx$. Essentially, the orbit of x is the set of elements in X that can be mapped through G.
- For some $x \in X$, the *stabilizer* of x (written as $Stab_G(x)$ or S_{xx}) is the set of all elements $g \in G$ such that $gx = x$. In other words, it is the set of elements in G with the (left) group action equal to x.
- The *transformer* of $x, y \in X$ (written as $trans(x, y)$ or S_{xy} is the set of all elements $g \in G$ such that $gx = y$.

With these definitions, we come to two important theorems.

Theorem 8.4 (Orbit-Stabilizer Theorem)**.** *Given any group action ϕ of a group G on a set X, all $x \in X$ satisfy*

$$|G| = |S_{xx}||O_x|.$$

Proof. First, we claim that for all $y \in O_x$, $|S_{xx}| = |S_{xy}|$. We will show a bijection between S_{xx} and S_{xy}. Take $g_{xx} \in S_{xx}$ and $g_{xy} \in S_{xy}$. Since

$$g_{xy}g_{xx}x = g_{xy}x = y.$$

This means that $g_{xy}g_{xx} \in S_{xy}$. This means $g_{xy}x = y$, so

$$g_{xy}^{-1}g_{xy}x = g_{xy}^{-1}y,$$

which implies

$$ex = x = g_{xy}^{-1}y.$$

This means $g_{xy}^{-1} \in S_{yx}$ and $g_{xy}^{-1}g_{xy} \in S_{xx}$.

Now, take any element $k \in S_{xy}$, which must exist since $y \in O_x$. We can establish a function

$$\chi : S_{xx} \to S_{xy} : g_{xx} \mapsto hg_{xx},$$

as well as a function

$$\sigma : S_{xy} \to S_{xx} : g_{xy} \mapsto k^{-1}g_{xy}.$$

Note that σ is the inverse of χ since

$$\chi(\sigma(g_{xy})) = \chi(k^{-1}g_{xy}) = kk^{-1}s_{xy} = g_{xy}$$

and

$$\phi(\chi(g_{xx})) = \phi(kg_{xx}) = k^{-1}kg_{xx} = g_{xx}.$$

Thus, χ is a bijection, as we claimed.

From here, the proof is direct. Since the group action is a function, G has set partitions $S_{xy} : y \in O_x$ by the definition of orbits. This implies

$$|G| = |S_{xx}||O_x|.$$

■

Now, we will prove Burnside's Lemma, which finds the number of orbits of a finite set acted on by a finite group. It is useful when counting objects with symmetry.

Theorem 8.5 (Burnside's Lemma)**.** *Let G be a finite group that acts on the set X and X/G be the set of orbits of X. For any element $g \in G$,*

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|$$

where X^g is the set of points of X which are fixed by g.

Proof. We start by changinging $\sum_{g \in G} |X^g|$ into a sum over X:

$$\begin{aligned}
\sum_{g \in G} |X^g| &= \sum_{g \in G} |x \in X : g \cdot x = x| \\
&= |(g, x) : g \in G, x \in X, g \cdot x = x| \\
&= \sum_{x \in X} |g \in G : g \cdot x = x| \\
&= \sum_{x \in X} |S_{xx}|.
\end{aligned}$$

We have $|S_{xx}| = \frac{|G|}{|O_x|}$ from the Orbit-Stabilizer Theorem, so the sum becomes

$$\begin{aligned}
\frac{1}{|G|} \sum_{x \in X} |S_{xx}| &= \frac{1}{|G|} \sum_{x \in X} \frac{|G|}{|O_x|} \\
&= \sum_{x \in X} \frac{1}{|O_x|}.
\end{aligned}$$

Each x in a given orbit O contributes $\frac{1}{|O|}$ to the sum, since there are $|O|$ of these, each orbit contributes 1. Thus, the sum is the number of orbits, so

$$|X/G| = \frac{1}{|G|} \sum_{x \in X} |X^g|.$$

■

Example. How many different ways are there to color the sides of a square by n colors, if two arrangements that can be obtained from each other by rotation are identical?

Solution Note that

- rotating by 0° fixes all n^4 elements,
- rotating by 90° fixes n elements,
- rotating by 180° fixes n^2 elements,
- rotating by 270° fixes n elements,

Since there are 4 transformations, we have

$$|X/G| = \frac{n^4 + n^2 + 2n}{4}.$$

We can directly plug in any number of colors we want to try, and we have avoided casework!

9. Polya's Enumeration Theorem

Theorem 9.1 (Polya's Enumeration Theorem (Unweighted))**.** *Let X be a set with group action induced by a permutation group G on X. Take C to be a set of colors on X and C^X to be the set of functions $f : X \to C$. Then,*

$$|C^x/G| = \frac{1}{|G|} \sum_{g \in G} |C|^{c(g)},$$

where we define $c(g)$ to be the number of cycles of g on X.

Proof. The f functions assign colors to the elements of X, and since $|C|^{c(g)}$ is the number of points fixed by g, this is equivalent to Burnside's Lemma, which has already been proven. ■

To tackle the weighted version, we need a few more definitions and concepts.

We denote C^D to be the set of all functions $f : X \to Y$, or the possible colorings of D where C is a finite set of colors and D is a finite set.

Definition 9.2 (Weight)**.** Let the colors $c \in C$ have positive integer weights $w(c)$. The weight of a coloring q is the sum of the weights of the colors:

$$w(q) = w(q(x)).$$

Definition 9.3 (Configuration Generating Function)**.** Let C be a set of all configurations c, where a configuration is an equivalence class with functions in the same orbit ϕ'. The CGF is defined

$$F(C) = \sum_{c \in C} W(c).$$

This standard notation $F(C)$ may be confused with that of EGF, so we will use $CGF(C)$ to denote it instead.

Finally, we can prove Polya's Enumeration Theorem.

Theorem 9.4 (Polya's Enumeration Theorem (Weighted))**.** *Let G be a group action on C^D where C^D is a set of colorings. w is a weight function on C. The CGF of C^D is equal to*

$$Z_G(\sum_{c \in C} w(c), \sum_{c \in C} w(c)^2, \sum_{c \in C} w(c)^3, \dots).$$

Proof. We can manipulate the CGF ($\sum_{c \in C} W(c)$) by summing it up over weights instead of configurations to get

$$\sum_{w} w|f \in F : W(f) = w|.$$

Since all the elements in the same configuration have the same weight, each configuration is in an orbit of X_w, which we can define to be the set of colorings with weight w.

Applying Burnside's Lemma yields

$$|f \in F : W(f) = w| = |X_w/G| = \frac{1}{|G|} \sum_{g \in G} |Fix(g)|,$$

where $|Fix(g)|$ is the number of fixed points g with weight w.

Thus, the CGF is equal to

$$\begin{aligned}\sum_{w} w \frac{1}{|G|} \sum_{g \in G} |Fix(g)| &= \frac{1}{|G|} \sum_{g \in G} \sum_{w} w|Fix(g)| \\ &= \frac{1}{|G|} \sum_{g \in G} \sum_{x \in Fix(g)} W(x).\end{aligned}$$

Since every $g \in G$ permutes the set X, every g splits X into a disjoint union of cycles. Each cycle of length m can be represented by the generating function $\sum_{c \in C} w(c)^m$, so

$$\frac{1}{|G|} \sum_{g \in G} \sum_{x \in Fix(g)} W(x) = \prod_{cycles} \sum_{c \in C} w(c)^m.$$

Finally, plugging this back into our expression for the CGF yields

$$CGF = \frac{1}{|G|} \sum_{g \in G} \prod_{cycles} \sum_{c \in C} w(c)^m = Z_G \left(\sum_{c \in C} w(c), \sum_{c \in C} w(c)^2, \sum_{c \in C} w(c)^3, \dots \right).$$

This is exactly what we wanted to show.

■

Note that if we let the weights all be 0, we have

$$|C^D/G| = CGF(0) = Z_G(m, m, \dots, m) = \frac{1}{|G|} \sum_{g \in G} |C|^{c(g)},$$

which is the unweighted version.

Some famous applications of Polya's Enumeration Theory are counting the number of isomers given a chemical formula, enumerating the number of beaded necklaces with n colors, and even generalizing number theory theorems like Fermat's Little Theorem!

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