Group Theory of the Rubik's Cube

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Interesting Questions

1. How many possible positions of the Rubik's Cube are there?
2. How many possible positions of the 2x2 Rubik's Cube group are there?
3. What is the minimum number of moves needed to solve a cube from any position?

Group Theory Concepts

1. Definition of a Group
2. Subgroups
3. Cyclic groups, generators
4. Permutations
5. Kernel
6. Direct products, semi-direct products, wreath products

Definition of a Group

A group is a set with a binary operation $*$. It also must satisfy these conditions:

1. $(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$ for all $g_1, g_2, g_3 \in G$
2. The group must have an element that maps all elements to themselves. $e * g = g$ and $g * e = g$ for all $g \in G$. This element is called the unit element, or the identity.
3. Each element in the group must have an inverse. An inverse maps an element to the unit element. $g * g^{-1} = e$, and $g^{-1} * g = e$.

Subgroup

Let H be a subset of a group G. Then H is a subgroup of G if H can be a group with the same operation as G. H is a normal subgroup of G if, for each $a \in G$, $a^{-1}Ha = H$

Permutations

The symmetric group, denoted S_n is the group of all possible permutations of n elements. The operation of this group is a composition of shuffling of the n elements.

The alternating group, denoted A_n, is a subgroup of S_n that only contains even permutations. An even permutation is a permutation that can be achieved through swapping two elements at a time an even number of times.

Cyclic Groups and Generators

$$G = \langle g_0 \rangle$$

G is a cyclic group with generator g_0, meaning all $g \in G$ can be written as $g = g_0^k$ for some $k \in \mathbb{Z}$.
Notation: C_n

Example

The group $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$ under addition modulo 5 is a cyclic group with generator 1, as each element can be written as 1 added to itself a certain amount of times.

Kernel

Let G_1 and G_2 be groups. Additionally, let there be a homomorphism $\phi : G_1 \to G_2$. The kernel, denoted $\ker(\phi)$, is the group of all elements in G_1 that map to the identity element of G_2.

Direct Product

The direct product between two groups G_1 and G_2, denoted

$$G_1 \times G_2,$$

is the group of all ordered pairs (g_1, g_2), $g_1 \in G_1$, $g_2 \in G_2$ under the operation $(g_1, g_2) * (g_1', g_2') = (g_1 * g_1', g_2 * g_2')$.

Semi-direct Product

The semi-direct product between two subgroups G_1 and G_2, denoted is

$$G_1 \rtimes G_2,$$

the group $A = G_1 G_2$ where $G_1 \cap G_2 = e_A$ and G_1 is a normal subgroup of A

Example

$S_n = A_n \rtimes C_2$

Wreath Product

Let G be a group, X be the finite set $\{1, 2, 3, \ldots t\}$, and H be a group acting on X. Let G^t denote the direct product of G with itself t times. Then the wreath product of G and H is

$$G^t \wr H := G^t \rtimes H,$$

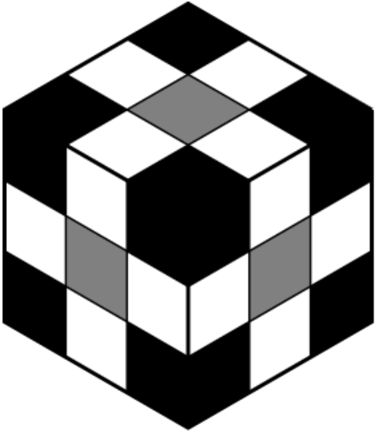
where H acts on G^t through its action on X.

The Rubik's Cube

1. Types of pieces
2. Defining move notation
3. The Illegal Group
4. Fundamental Theorem of Cube Theory
5. The Legal Group
6. The minimum number of moves needed to solve any position

Types of Pieces

Figure 2. Edges, Corners, and Centers



Notation

R: Right
L: Left
U: Up
D: Down
F: Front
B: Back

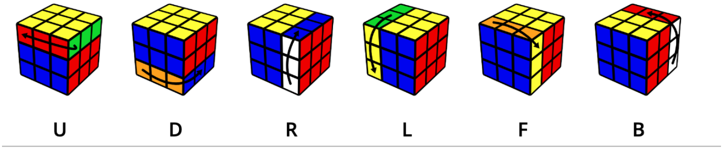


Figure: Standard Moves

Image Credit: https://jperm.net/3x3/moves

Slice Moves

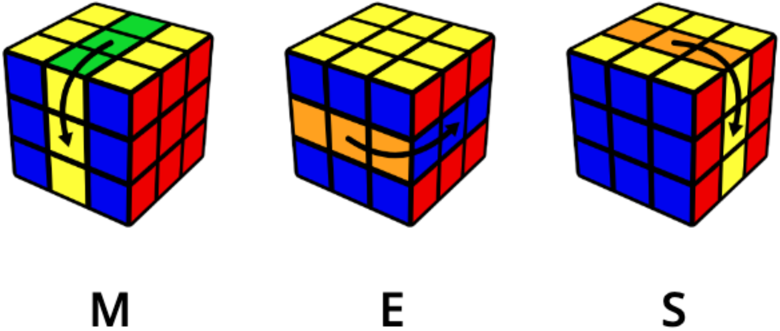
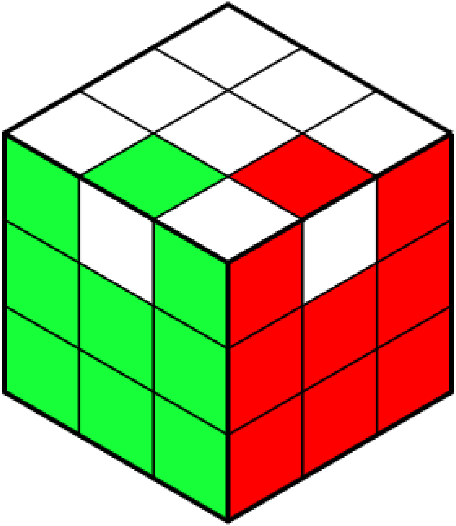


Image Credit: https://jperm.net/3x3/moves

Illegal and Legal Rubik's Cube Group

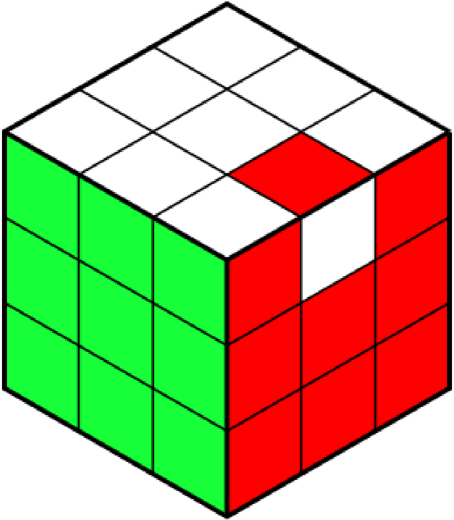
Illegal Group: $I = (C_2^{12} \wr S_{12}) \times (C_3^8 \wr S_8)$
Legal Group: We need more rules first

Solvable Position



$$R^{-1}FRU^{-1}M^{-1}U^{2}MU^{-1}SR^{-1}F^{-1}RS^{-1}$$

Unsolvable Position



First Fundamental Theorem of Cube Theory

Theorem (First Fundamental Theorem of Cube Theory)

Let $v \in C_3^8$, $r \in S_8$, $w \in C_2^{12}$, *and* $s \in S_{12}$ *be the four variables that describe a position of the cube. The position is achievable solely through turns if and only if:*

1. $sgn(s) = sgn(r)$
2. $w_1 + w_2 + w_3 + \ldots + w_{12} = 0 \pmod 2$
3. $v_1 + v_2 + v_3 + \ldots + v_8 = 0 \pmod 3$

Legal Rubik's Cube Group

Intermediate Step: $G_0 = (C_2^{11} \rtimes S_{12}) \times (C_3^7 \rtimes S_8)$

Let (v, r, w, s), $w \in C_2^{11}, s \in S_{12}, v \in C_3^7, r \in S_8$ denote a position of the cube.
Let's define $\phi : G_0 \to \{-1, 1\}$ such that

$$\phi(v, r, w, s) = sgn(r)sgn(s)$$

Legal Rubik's Cube Group: $G = \ker(\phi)$

Order:

$$\begin{aligned} |G| &= 1/2 * |G_0| = 2^{10} * 12! * 3^7 * 8! \\ &= 43{,}252{,}003{,}274{,}489{,}856{,}000 \end{aligned}$$

Minimum Moves

- 20 moves (half-turn metric)
- 26 moves (quarter-turn metric)

$4.3 * 10^{19}$ doesn't fit on a computer

$$H = \langle U, D, R2, F2, L2, B2 \rangle$$

$|H| = 19,508,428,800$
$2,217,093,120$ cosets

Further Questions

1. What is the Illegal and Legal Rubik's Cube Group for other types of Rubik's Cubes such as the pyraminx, megaminx, etc.
2. How can we figure out the minimum number of moves between two positions of the cube?